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CONTENTS

Original scientific papers

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Assessing the pure technical efficiency of public primary schools in the City of Zagreb

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Abstract. This paper is the first to assess the pure technical efficiency (PTE) of 111 public primary schools in the City of Zagreb during the 2022/2023 school year, using an input-oriented Data Envelopment Analysis (DEA) – BCC model. PTE was measured using two input variables (expenditures and number of teachers) and three output variables (average grades, number of pupils who passed the class, and secondary school enrolment points). The analysis revealed substantial variation in efficiency scores, with 20% of schools identified as fully efficient. The average efficiency score was 0.90, suggesting that, on average, schools could reduce inputs by 10% while maintaining current output levels. A follow-up super-efficiency DEA model was applied to rank efficient schools and identify best-practice examples that could serve as models for others. Based on super-efficiency scores, schools were grouped into five performance categories – best practice, efficient, near-efficient, emerging, and low efficiency. Additional findings suggest that larger schools tend to be more efficient and that benchmarking within peer-size groups can help uncover high-performing schools that might otherwise remain under the radar. The results underscore the importance of context-sensitive efficiency evaluation in education and highlight the need for improved data on school characteristics to support more comprehensive, actionable assessments.

Keywords: City of Zagreb, Data Envelopment Analysis, public primary schools, pure technical efficiency, super-efficiency

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Original scientific paper.

1. Introduction

The growing number of studies on the efficiency of public services can be seen as a consequence of the increasing demand for more and better public services worldwide. The most visible to and important for each citizen, it is the efficiency of public services at the local level of government that has been the most researched (e.g., [30], [31], [35]).

According to the theory of fiscal federalism, lower levels of government are more capable of delivering public services to citizens because they have a greater understanding of the needs of their inhabitants [42]. Similarly, Seabright [38] stresses that the local authorities' physical closeness to the end users fosters local authority accountability and improves public service delivery because of better electoral controls. On the other hand, service delivery at the local government level may impair efficiency if, for instance, there is limited managerial capacity, the service relies on economies of scale, or if influential interest groups exert undue control over local decision-making [26].

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The efficiency of local public services refers to the extent to which local governments deliver services to citizens, minimising use of resources while meeting public needs. The resources necessary for the provision of public services are input variables (inputs). The services provided are treated as output variables (outputs). Therefore, an input-output analysis can be used to assess the efficiency of local governments – that is, their ability to deliver a greater quantity and quality of public services using limited resources [26]. Results from various countries consistently highlight significant disparities in efficiency scores among local governments, indicating the existence of substantial room for improvement in the delivery of public services (e.g., [6], [33]).

The efficiency of public primary schools in the City of Zagreb has not been previously assessed. Given the vital role primary education plays in shaping the future of pupils, families, and society at large – and considering the limited resources within city budgets – it is essential to identify which schools are able to optimize resource use while maintaining strong educational outcomes. This study employs the deterministic nonparametric frontier method – DEA – to evaluate the efficiency of 111 primary schools in the City of Zagreb during the 2022/2023 school year. We identified 20% of schools as fully efficient, with an average efficiency score of 0.90 across the sample. The super-efficiency DEA analysis allowed for the ranking of efficient schools and the identification of five best-practice schools. Additional findings suggest that larger schools generally demonstrate higher efficiency, and that benchmarking within peer-size groups can reveal high-performing schools that might otherwise go unnoticed. This analysis serves as an initial step toward identifying which primary schools in the City of Zagreb could improve their efficiency in terms of the criteria used in the study. National and local governments, as well as researchers, should give particular attention to schools with low efficiency levels and undertake further analysis to uncover opportunities for improvement.

The structure of the paper is as follows: Section 2 reviews the literature on efficiency measurement, with a focus on public services, education, and relevant findings from the Croatian context. Section 3 describes methodology, context, and the data used in the analysis. Section 4 presents the results and discussion, while Section 5 offers a conclusion.

2. Literature review

Defining efficiency in public services is inherently complex. Farrell [18] distinguishes between two key components: technical efficiency, referring to the ability to maximise output from a given set of inputs, and allocative efficiency, which concerns using inputs in optimal proportions to minimise cost. Expanding on this, Andrews and Entwistle [3], propose a broader framework, including distributive efficiency – the equitable allocation of services among different social groups – and dynamic efficiency, which considers the balance between current and future public spending. While these definitions vary, technical efficiency remains the most widely applied in empirical research, particularly due to its compatibility with quantifiable input-output relationships. As noted by Narbón-Perpiñá and De Witte [30, 31], technical efficiency (the ability to maximise output from a given set of inputs) captures the cost-effectiveness and quality of services delivered and is the definition adopted in this study. Efficiency assessments typically rely on either parametric or non-parametric methods. Among parametric models, stochastic frontier analysis (SFA) allows for random error in efficiency estimation and has been used in studies such as Grosskopf et al. [22] and Conroy and Arguea [11]. Non-parametric methods, particularly DEA, have become increasingly prevalent due to their flexibility and ability to handle multiple inputs and outputs without imposing a functional form [9]. Another non-parametric approach, Convex Nonparametric Least Squares (CNLS), has also been applied in efficiency analysis. Some scholars have combined both approaches for greater robustness (e.g., [13]), while De Witte and López-Torres [14] provide an extensive review of these methodologies in the education sector. DEA is also used for ranking by Dobos [16] in various areas. For example, Sinha et al. [40] used CNLS to evaluate the efficiency of Indian general insurance companies, demonstrating the method's potential for application in different sectors, Kumari et al. [24] applied DEA to the stock market. Models of DEA continues to evaluate, e.g. inverse DEA [21]. The modelling of educational efficiency often distinguishes between input-oriented and output-oriented perspectives. While input-oriented models emphasise minimising resources to achieve a given level of output, output-oriented models seek to maximise outcomes with the available inputs.

In the education sector, schools are commonly treated as decision-making units (DMUs) that convert resources into educational outcomes. Measuring their efficiency poses challenges, particularly in defining appropriate output indicators. Primary education aims not only at academic achievement but also at fostering behavioural, emotional, and social development. As a result, outputs may include test scores, graduation or promotion rates, pupil satisfaction, or engagement in extracurricular activities ([37], [19]). Input variables typically include school expenditures, number of teachers, teacher qualifications, class size, and infrastructure ([15], [20]). Due to data limitations, many studies use proxies and tailor their variables to the context and availability of information.

In decentralised systems, several studies have assessed educational efficiency using local governments as DMUs. Inputs commonly include public expenditures on education, while outputs reflect the scale of service provision – most often the number of enrolled pupils, interpreted as a proxy for demand [35]. Other studies adopt alternative proxies for education services provided, such as the number of schools (e.g., [34]) or the number of classes per school [32]. Sampaio de Sousa and Stošić [36] propose more nuanced output indicators, including enrollment and attendance rates, grade progression, and age-appropriate placement, aiming to reflect both the quantitative and qualitative dimensions of service delivery.

Although DEA has not yet been applied to assess primary education efficiency in Croatia, several studies have explored the factors influencing educational success at the primary school level. Babarović et al. [4] emphasise that pupils' cognitive abilities are the most significant predictors of academic outcomes, followed by family background characteristics such as parental education and household income. A nationwide study by Burušić et al. [8], involving 842 primary schools, quantified these influences: pupil-related factors explained between 5.3% and 15.9% of the variance in achievement across subjects; teacher-related variables (e.g., gender, qualifications, continuity) added 0.2% to 1.1%, while school-level characteristics (e.g., size, governance, support staff, principal's tenure) accounted for an additional 0.2% to 0.8%. These findings highlight the importance of multi-layered contextual factors in modelling school performance.

The literature review reveals three compelling reasons to examine primary education in Croatia. First, existing research has highlighted persistent inefficiencies in Croatia's public education spending compared to other EU member states (e.g., [41]). Second, existing research on the efficiency of public primary education in Croatia remains limited. Third, to date, no study has applied DEA or a similar method to evaluate the efficiency of primary education in the Croatian context. To address this research gap, we focus on all primary schools established by the City of Zagreb – the country's capital and largest city – as a starting point for such analysis.

3. Methodology, context, and data

3.1. Methodology

We use DEA to assess the PTE of primary schools. Analysing PTE allows us to assess how efficiently a school utilises its available resources to achieve optimal educational outcomes. Each school is treated as a DMU, meaning it is evaluated as part of a group that uses similar inputs to produce comparable outputs. DEA establishes an empirical efficiency frontier, where a score of

1 indicates full efficiency. This frontier is defined by minimizing inputs and maximizing outputs. The efficiency frontier serves as an attainable benchmark that inefficient DMUs should aim to reach. Any deviation from this frontier, as determined by the deterministic DEA method, is considered inefficiency and may reflect the influence of random external factors, such as institutional and socioeconomic conditions or measurement errors [10].

The most commonly used basic DEA models are: (1) the Charnes, Cooper and Rhodes [9] model, which assumes that the production function shows a constant return to scale (CCR model), and (2) the Banker, Charnes and Cooper [5] model which assumes a variable return to scale (BCC model). The choice of model depends on the content of the analysis and whether a short- or a long-term analysis is concerned. The CCR model primarily focuses on measuring scale efficiency, aiming to evaluate whether a school's size – defined by factors such as resources and infrastructure – is optimised to achieve the best possible educational outcomes. The BCC model measures PTE, i.e., how efficiently a unit utilises its available resources (inputs) to produce outputs, focusing solely on its management and operational practices, without accounting for the scale of operations (i.e., whether the unit is operating at an optimal size). BCC assumes variable returns where a proportional increase in input results in more or less than a proportional increase in output.

Following Cooper et al. [12], if a set of n DMUs is considered ($DMU_j, j = 1, \ldots, n$), each of them produces s outputs using m inputs. Let the $x_j = \{x_{ij}, i = 1, \ldots, m\}$ represent the input vector and $y_j = \{y_{rj}, r = 1, \ldots, s\}$ the output vector of DMU_j. The data set is determined by input matrix $X = (x_{ij}, i = 1, \ldots, m, j = 1, \ldots, n)$ and output matrix $Y = (y_{rj}, r = 1, \ldots, s, j = 1, \ldots, n)$. The fundamental aim of the DEA model is the evaluation of DMU efficiency. The vectors $X\lambda$ and $Y\lambda$, $\lambda = (\lambda_1, \ldots, \lambda_n), \lambda > 0$ represents the proportions contributed by efficient DMUs to the DMU projections onto the efficient frontier, and e is the unit vector.

In our paper, we employ the BCC input-oriented model. Seeking the efficiency score of the DMU could be formulated as a standard linear programming model. The BCC input-oriented model for evaluation of the DMU_0 that will be applied in our paper can be formulated as follows:

$$\max_{\theta\lambda} \theta_0$$

subject to (1)

$$\begin{aligned} -y_0 + Y\lambda &\geq 0, \\ \theta x_0 - X\lambda &\geq 0, \\ e^T\lambda &= 1, \\ \lambda &\geq 0, \end{aligned}$$

where x_0 and y_0 are vectors of inputs and outputs of the unit under evaluation, θ_0 is the BCC efficiency score of the DMU_0, and e^T is the unit row vector. The goal is to achieve assigned weights that maximise the ratio for the particular DMU being analysed. Due to the setting of real constraints, the optimal value is 1. An input-oriented DEA model aims to assess how much input quantities can be proportionally reduced while maintaining the same output level. If a unit is inefficient, it suggests there is potential to improve operations, such as through better resource management, to achieve the same output with fewer inputs. This approach is particularly relevant for public institutions facing pressures to reduce costs while maintaining a certain level of services [6], which is why we have opted for the input-oriented approach. This study adopts a variable returns to scale model, based on the assumption that schools may be suboptimal in producing public services due to constraints, such as budget limitations [35].

3.2. Context and data

Primary education in Croatia, typically lasting eight years, is both compulsory and provided free of charge. Its objectives and responsibilities are defined by curricula set out by the Ministry of Science, Education, and Youth (MSEY). Although primary schools can be established by national, regional, and local authorities, as well as other legal entities and individuals, our analysis specifically focuses on public schools established by the City of Zagreb. This choice was driven by their considerable number, consistency in organisational structure and funding, and the greater accessibility of relevant data. Additionally, focusing on this homogeneous group enhances the comparability of results and ensures greater policy relevance at the city level. We exclude specialised public primary schools for pupils with disabilities (e.g., Center for Autism, Goljak Education Center, Suvag Polyclinic, and Nad Lipom Primary School) due to significant differences in their curricula, resources, and educational objectives, making comparisons with regular schools unsuitable.

Public primary schools in Croatia are funded by the central government, counties, and cities. However, the majority of the funding comes from the central government, primarily through MSEY. This funding covers teachers' salaries and professional development, education for children with developmental difficulties, gifted pupils, and ethnic minorities, as well as initiatives related to digitalization and library resources (Eurydice Network [17]). Counties and cities, as the governing authorities of public primary schools, are responsible for financing the so-called decentralised functions. These include material costs, maintenance and investment in infrastructure, equipment and teaching aids, pupil transportation, capital construction in line with standards set by the central government, co-financing of extended and full-day programs, and the provision of school meals (Eurydice Network [17]). Each year, the central government sets a minimum financial standard for these decentralised functions. Funding is primarily secured through an increased share of personal income tax allocated to counties and cities, supplemented by equalisation grants in cases where these local governments cannot meet the standard solely through tax revenue. Counties and cities may also provide additional funding from their own budgets; for instance, the City of Zagreb has for many years provided free textbooks to all primary school pupils.

Our research analyses 111 primary schools in the City of Zagreb in the 2022/2023 school year. This year was selected as it is the most recent period for which comprehensive data are available. Drawing on available data and existing literature, we selected two input variables at the school level (see Table 1): total annual expenditures, representing overall resource input (similar to [35]), and the number of teachers employed at each school, representing human resource input (as in [36]). We considered three output variables at the primary school level: average pupil grades, the number of pupils successfully passing the class, and the average points pupils achieve when enrolling in secondary schools. These outputs capture distinct dimensions of school performance. Specifically, average grades reflect the quality of teaching and learning, passing rates indicate operational efficiency in delivering the curriculum, and secondary school enrolment points show how effectively schools prepare pupils for subsequent educational stages – a critical outcome of primary education. Previous studies have employed similar output variables: average pupil grades (e.g., [7]), the number of pupils passing classes (e.g., [36], [7]), and secondary school enrolment points (e.g., [1], at the national level). We expect that primary schools with higher average pupil grades, higher pass rates, and higher average secondary school enrolment scores achieve superior educational outcomes, both in terms of quality and quantity.

Variable		**Definition and measurement**	**Source**
		Inputs	
Expenditures	x_{1j}	Total expenditures are based on data from school financial reports.	MF [27]
Teachers	x_{2j}	The number of teachers as of November 1, 2022.**	MSEY [28]
		Outputs	
Average grades	y_{1j}	The average pupil grades at the end of the school year.	MSEY[29]
Pupils passed	y_{2j}	The total number of pupils who pass the class, i.e., the number of pupils who receive a passing grade.	MSEY[29]
Enrolment points	y_{3j}	The average number of points achieved by pupils who complete eight years of primary school for enrolment into secondary school, based on their performance in the summer and fall terms.***	MSEY[29]

Table 1: *Definitions of variables used for public primary schools**

* All primary schools may have branch schools, which are included in our analysis.

** Since data on the number of teachers can vary significantly from day to day due to frequent replacements and other factors, using figures for the entire school year would result in unrealistically high numbers. Therefore, the situation as of November 1, 2022, was used to obtain a more accurate and realistic number of teachers for the 2022/2023 school year.

*** The number of points that Croatian primary school pupils achieve when enroling in secondary school is mostly based on their final school achievement from 5th to 8th grade and the school achievement in a specific set of subjects from 7th and 8th grades (e.g., Croatian, Mathematics). Pupils can also earn points based on criteria established through knowledge competitions and school sports club competitions. Finally, an additional few points can be awarded according to special criteria, including pupils with health problems, those living in difficult economic, social, and educational conditions, Roma pupils, and pupils whose parents are civil servants working on behalf of the Republic of Croatia abroad.

Note: The variable expenditures refers to the average values for 2022 and 2023; while the variables teachers, average grades, pupils passed, and enrolment points correspond to the 2022/2023 school year. Data on both output and input variables are available upon request.

Source: Authors.

4. Results and discussion

The descriptive statistics in Table 2 for the 2022/2023 school year reveal significant differences in the values of the variables, particularly in expenditures, number of teachers, and number of pupils passed. Matija Gubec Primary School has the highest total expenditures (3,133,744 EUR), while Tituš Brezovački Primary School has the highest number of teachers (89). In contrast, Savski Gaj Primary School has the highest number of pupils passed (1,139). Stjepan Benceković–Horvati Primary School reports the lowest expenditures (541,028 EUR), the lowest number of teachers (20), and the lowest number of pupils who pass the class (126). Dr. Vinko Žganec Primary School has the lowest average grade for all pupils at the end of the school year (4.09), while Matija Gubec Primary School has the highest (4.89). Finally, the highest average number of points achieved by 8th-grade pupils for enrolment into secondary school is found at August Harambašić Primary School (84.45), while the lowest is at Žitnjak Primary School (50.67).

Variable	**Min**	**Mean**	**Median**	**Max**	**Standard deviation**
Inputs					
Expenditures	541,028	1,791,060	1,788,155	3,133,744	46,897.27
Teachers	20	54	54.30	89	1.43
Outputs					
Average grades	4.09	4.68	4.65	4.89	0.01
Pupils passed	126	542	557.5	1,139	20
Enrolment points	50.67	71.40	70.09	84.45	0.79

Table 2: *Descriptive statistics for public primary schools*
Source: Authors.

Our analysis employs model (1), which measures PTE – that is, how efficiently a school utilises its resources (inputs) to generate outputs – focusing exclusively on internal management and operational practices. Moreover, several authors point out that this approach is the most commonly used and considered reliable for minimizing the risk of model misspecification [39]. The results of the efficiency analysis are presented in Table 3. The initial DEA model (1) identified an efficiency frontier comprising 20% of the schools in the sample, each achieving a full efficiency score of 1. The average PTE scores across the entire sample, as measured by the initial DEA model (1), was 0.90. This indicates that, on average, schools could reduce input usage by approximately 10% while maintaining the same output level. Building on the initial DEA model, we conducted a supplementary super-efficiency analysis to make further distinctions among the high-performing schools. Super-efficiency scores were calculated by removing the evaluated DMU (achieving a full efficiency in model (1)) from the reference set, allowing efficiency values to exceed 1 and enabling full ranking of efficient units [2]. This additional analysis enabled a more precise ranking of the efficient schools and facilitated the identification of benchmark units, thus providing valuable insights into best practices within the system of primary schools in the City of Zagreb.

Based on the distribution of super-efficiency scores and the identification of leading units, schools were grouped into five performance categories to allow a more refined interpretation of efficiency levels. The categories are defined as follows:

1. *Best practice*: A selected group of highly efficient schools identified through super-efficiency analysis as benchmarks. These schools not only score above 1 but also serve as reference points for others due to their superior performance.
2. *Efficient*: Schools with a super-efficiency score greater than or equal to 1 but not included in the best practice group. These schools operate efficiently but are not among the top-performing benchmarks.
3. *Near-efficient*: Schools with scores between 0.95 and 1 (excluding 1), indicating performance close to the efficiency frontier with little room for improvement.
4. *Emerging*: Schools with scores between 0.80 and 0.95 (excluding 0.95), showing moderate deviation from the efficiency frontier and higher potential for improvement.
5. *Low efficiency*: Schools with scores below 0.80, reflecting the considerable room for performance improvement.

This classification provides a clearer view of relative efficiency levels and supports the identification of top-performing schools as well as those with development management needs (Table 3). While the DEA identifies schools that are relatively efficient within the sample, it is important to note that this does not necessarily mean a school is fully optimised in an absolute sense. DEA evaluates each school's performance in relation to others in the same dataset, rather than against a theoretical maximum [23]. In other words, efficiency scores reflect how well a school performs compared to its peers using the same set of resources, rather than whether it operates at a universally optimal level.

Primary school	PTE	Super-efficiency	Rank	Category
August Harambašić	1	Benchmark	1	Best practice
Bartol Kašić	1	Benchmark	2	Best practice
Savski Gaj	1	Benchmark	3	Best practice
Tituš Brezovački	1	Benchmark	4	Best practice
Matija Gubec	1	Benchmark	5	Best practice
Stjepan Benceković-Horvati	1	1.5295	6	Efficient
Središće	1	1.5147	7	Efficient
Remete	1	1.3981	8	Efficient
Trnjanska	1	1.2743	9	Efficient
Ivan Cankar	1	1.2167	10	Efficient
Jabukovac	1	1.1633	11	Efficient
Alojzije Stepinec	1	1.0960	12	Efficient
Ksaver Šandor Gjalski	1	1.0799	13	Efficient
Pavlek Miškina	1	1.0655	14	Efficient
Ivan Filipović	1	1.0515	15	Efficient
Granešina	1	1.0507	16	Efficient
Marin Držić	1	1.0472	17	Efficient
Josip Juraj Strossmayer	1	1.0445	18	Efficient
Dragutin Tadijanović	1	1.0281	19	Efficient
Gračani	1	1.0252	20	Efficient
Antun Gustav Matoš	1	1.0046	21	Efficient
August Šenoe	1	1.0027	22	Efficient
Vjenceslav Novak	0.9979	0.9979	23	Near-efficient
Izidor Kršnjavi	0.9979	0.9979	24	Near-efficient
Većeslav Holjevac	0.9911	0.9911	25	Near-efficient
Grof Janko Drašković	0.9898	0.9898	26	Near-efficient
Antun Branko Šimić	0.9864	0.9864	27	Near-efficient
Dragutin Domjanić	0.9781	0.9781	28	Near-efficient
Brestje-Sesvete	0.9718	0.9718	29	Near-efficient
Čučerje	0.9684	0.9684	30	Near-efficient
Josip Račić	0.9661	0.9661	31	Near-efficient
Horvati	0.9630	0.9630	32	Near-efficient
Brezovica	0.9627	0.9627	33	Near-efficient
Cvjetno Naselje	0.9619	0.9619	34	Near-efficient
Dragutin Kušlan	0.9614	0.9614	35	Near-efficient
Bukovac	0.9585	0.9585	36	Near-efficient
Dr. Ante Starčević	0.9535	0.9535	37	Near-efficient
Šestine	0.9524	0.9524	38	Near-efficient
Retkovec	0.9514	0.9514	39	Near-efficient
Medvedgrad	0.9511	0.9511	40	Near-efficient
Vladimir Nazor	0.9490	0.9490	41	Emerging
Jure Kaštelan	0.9484	0.9484	42	Emerging
Braće Radić	0.9471	0.9471	43	Emerging
Pantovčak	0.9466	0.9466	44	Emerging
Lučko	0.9449	0.9449	45	Emerging
Luka-Sesvete	0.9420	0.9420	46	Emerging
Ljubljanica	0.9403	0.9403	47	Emerging

Primary school	PTE	Super-efficiency	Rank	Category
Rapska	0.9395	0.9395	48	Emerging
Ivanja Reka	0.9376	0.9376	49	Emerging
Ivan Mažuranić	0.9358	0.9358	50	Emerging
Stenjevec	0.9319	0.9319	51	Emerging
Ante Kovačić	0.9294	0.9294	52	Emerging
Vukomerec	0.9290	0.9290	53	Emerging
Sesvete	0.9271	0.9271	54	Emerging
Malešnica	0.9260	0.9260	55	Emerging
Borovje	0.9257	0.9257	56	Emerging
Žuti Brijeg	0.9241	0.9241	57	Emerging
Julije Klović	0.9233	0.9233	58	Emerging
Tin Ujević	0.9228	0.9228	59	Emerging
Sveta Klara	0.9219	0.9219	60	Emerging
Odra	0.9079	0.9079	61	Emerging
Vrbani	0.9073	0.9073	62	Emerging
Oton Iveković	0.9066	0.9066	63	Emerging
Silvije Strahimir Kranjčević	0.9064	0.9064	64	Emerging
Voltino	0.8968	0.8968	65	Emerging
Markuševec	0.8963	0.8963	66	Emerging
Sesvetska Sela	0.8960	0.8960	67	Emerging
Kralj Tomislav	0.8945	0.8945	68	Emerging
Matko Laginja	0.8941	0.8941	69	Emerging
Otok	0.8869	0.8869	70	Emerging
Špansko Oranice	0.8792	0.8792	71	Emerging
Kustošija	0.8789	0.8789	72	Emerging
Jelkovec-Sesvete	0.8784	0.8784	73	Emerging
Marija Jurić Zagorka	0.8767	0.8767	74	Emerging
Iver-Sesvetski Kraljevec	0.8752	0.8752	75	Emerging
Ivan Granđa-Solblinec	0.8696	0.8696	76	Emerging
Sesvetski Kraljevec	0.8614	0.8614	77	Emerging
Rudeš	0.8586	0.8586	78	Emerging
Mato Lovrak	0.8583	0.8583	79	Emerging
Ban Josip Jelačić	0.8572	0.8572	80	Emerging
Fran Galović	0.8563	0.8563	81	Emerging
Dobriša Cesarić	0.8541	0.8541	82	Emerging
Trnsko	0.8534	0.8534	83	Emerging
Mladost	0.8519	0.8519	84	Emerging
Lovro Pl. Matačić	0.8491	0.8491	85	Emerging
Zapruđe	0.8461	0.8461	86	Emerging
I. primary school Dugave	0.8393	0.8393	87	Emerging
August Cesarec	0.8377	0.8377	88	Emerging
Sesvetska Sopnica	0.8347	0.8347	89	Emerging
Nikola Tesla	0.8333	0.8333	90	Emerging
Gornje Vrapče	0.8319	0.8319	91	Emerging
Hrvatski Leskovac	0.8314	0.8314	92	Emerging
Dr. Vinko Žganec	0.8305	0.8305	93	Emerging
Ivan Meštrović	0.8282	0.8282	94	Emerging

Primary school	PTE	Super-efficiency	Rank	Category
Gustav Krklec	0.8145	0.8145	95	Emerging
Ivan Gundulić	0.8054	0.8054	96	Emerging
Petar Preradović	0.8025	0.8025	97	Emerging
Kajzerica	0.8014	0.8014	98	Emerging
Miroslav Krleža	0.8010	0.8010	99	Emerging
Davorin Trstenjak	0.7978	0.7978	100	Low efficiency
Žitnjak	0.7938	0.7938	101	Low efficiency
Dr. Ivan Merz	0.7719	0.7719	102	Low efficiency
Ivan Goran Kovačić	0.7645	0.7645	103	Low efficiency
Petar Zrinski	0.7572	0.7572	104	Low efficiency
Prečko	0.7517	0.7517	105	Low efficiency
Antun Mihanović	0.7466	0.7466	106	Low efficiency
Fran Krsto Frankopan	0.7443	0.7443	107	Low efficiency
Ivo Andrić	0.7306	0.7306	108	Low efficiency
Vugrovec-Kašina	0.7170	0.7170	109	Low efficiency
Jordanovac	0.6798	0.6798	110	Low efficiency
Grigor Vitez	0.5305	0.5305	111	Low efficiency

Table 3: *DEA PTE and super-efficiency results with ranking and category assignment for 111 primary schools in the City of Zagreb*
Source: Authors.

Best practice units – also referred to as benchmark units – are those that not only achieve full efficiency but also serve as reference points in the construction of the efficiency frontier ([12]). These schools are used as comparators for less efficient schools and play a key role in defining what is considered optimal input-output performance. In the super-efficiency model, best practice schools are identified by their ability to maintain or exceed an efficiency score of 1 even when they are excluded from the reference set. This indicates that they outperform other schools without relying on themselves as benchmarks [25]. In our super-efficiency DEA analysis, the schools identified as benchmarks are August Harambašić, Bartol Kašić, Savski Gaj, Tituš Brezovački, and Matija Gubec. These schools not only demonstrate optimal performance relative to their peers but also serve as exemplary models within the system. They represent a tangible standard of excellence in both resource utilisation and educational outcomes and can play a key role by sharing efficient practices with other schools.

Since Burušić et al. [8] identified primary school size as a significant predictor of pupils' academic success in Croatia; we also sought to examine whether schools with more pupils tend to achieve higher efficiency scores. Interestingly, in the initial model (1), the average number of pupils in efficient schools (score = 1) is almost identical to that of inefficient ones (565 vs. 557), suggesting that there is no clear relationship between school size and efficiency at this level. However, a much larger difference emerges when examining the best practice schools identified through the super-efficiency analysis. These schools have an average of 808 pupils, compared to 547 in the remaining schools. This contrast suggests that while size alone may not determine whether a school is efficient per se, it may influence the likelihood of becoming a top-performing benchmark. In addition, among the best practice schools, two – Savski Gaj and Tituš Brezovački – have the highest number of pupils in the entire sample, while four of the five exceed the median pupil count (all except August Harambašić).

To explore whether school size influences efficiency outcomes and to minimise potential bias in the efficiency assessment, an additional analysis was conducted. Schools were divided into two groups based on the median number of pupils: lower-capacity schools (below the median) and higher-capacity schools (above the median). For each group, average efficiency scores were calculated using the initial DEA model (1), without applying the super-efficiency extension (see Table 4). This approach ensured that all schools, including those on the efficiency frontier (score = 1), remained within the reference set, thereby preserving the comparability and integrity of the sample. Full results and rankings from this analysis are available upon request from the corresponding author.

School group	**Average efficiency (full sample)**	**Average efficiency (within their group)**
higher-capacity (above median)	0.92	0.94
lower-capacity (below median)	0.88	0.92

Table 4: *Average efficiency scores by school capacity group, model (1)*
Source: Authors.

Table 4 highlights a clear distinction in average efficiency scores between the two groups. In the full sample, the group of higher-capacity schools achieved a higher average efficiency score (0.92) than that of lower-capacity schools (0.88). This difference persisted even when efficiency was calculated within each group separately, with higher-capacity schools again outperforming lower-capacity ones (0.94 vs. 0.92). These findings suggest a moderate scale effect, where larger schools tend to operate more efficiently on average. While the gap is not large, its consistency across both the full sample and group-specific analyses indicates that school size may confer certain advantages in terms of input utilisation and output performance. Our findings align with those of Babarović et al. [4], who, in a study of 842 primary schools in Croatia, found that pupils in smaller schools tend to exhibit lower average academic performance. They attributed this outcome to the specific operational characteristics of smaller schools, while also noting that it may reflect structural features of Croatia's primary education system. This reinforces our earlier observation that best practice schools tend to have higher pupil numbers and underscores the importance of considering school size in both policy recommendations and the interpretation of DEA-based efficiency results.

Moreover, after dividing the sample into two groups based on the median number of pupils, four of the original five best practice schools retained their benchmark status within the higher-capacity group (Savski Gaj, Tituš Brezovački, Bartol Kašić, and Matija Gubec), while the one school below the median (August Harambašić) remained a benchmark within the lower-capacity group. In addition, five new benchmark schools emerged within the lower-capacity group – Jabukovac, Kustošija, Ivan Cankar, Izidor Kršnjavi, and Jordanovac – which had not been recognised as such in the full-sample analysis. These findings suggest that conducting efficiency analysis within more homogeneous groups allows for the identification of additional efficient units that may otherwise be underestimated when compared to significantly larger peers. In other words, smaller schools sometimes appear less efficient when compared with larger peers, although within their own peer group, they may demonstrate high performance.

5. Conclusion

This paper assessed the PTE of 111 public primary schools in the City of Zagreb in the 2022/2023 school year, using an input-oriented DEA – BCC model. The analysis aimed to evaluate how efficiently schools utilise available resources – measured by expenditures and the number of teachers – to generate key educational outcomes, including pupil achievement (average grades), academic progression (number of pupils passed), and preparedness for secondary education (secondary school enrolment points). The initial DEA model (1) identified 20% of schools as fully efficient, indicating a considerable share of schools already operating at the efficiency frontier. However, the average efficiency score of 0.90 in the full sample suggests that most schools could achieve the same level of output with approximately 10% fewer inputs. To further differentiate among efficient units, a super-efficiency DEA analysis was applied, which enabled the ranking of efficient schools and the identification of five best practice public primary schools that serve as benchmarks: August Harambašić, Bartol Kašić, Savski Gaj, Tituš Brezovački, and Matija Gubec. In addition to individual efficiency scores, schools were grouped into five performance categories – best practice, efficient, near-efficient, emerging, and low efficiency – based on the results of the super-efficiency analysis. This classification provided a more nuanced understanding of performance levels across schools and allowed for the identification of both high-performing units and those with greater potential for improvement.

The results also point to a possible impact of size on efficiency outcomes. While the initial DEA model showed almost no difference in average pupil numbers between efficient and inefficient schools, in super-efficiency analysis, best practice schools had a considerably higher average number of pupils, suggesting that school size may be associated with top-level efficiency. Additional analysis by capacity groups confirmed this finding: higher-capacity primary schools (those with pupil numbers above the median) achieved slightly higher average efficiency scores (0.92 vs. 0.88) in the full sample, a trend that remained consistent even within group-specific evaluations.

Furthermore, when schools were analysed within more homogeneous, size-based groups, additional efficient schools – including some smaller ones – emerged as benchmarks. This suggests that smaller schools may appear less efficient when directly compared to much larger peers, despite performing strongly within their own context. These findings underscore the importance of adjusting efficiency analyses to take account of structural characteristics, such as school size, highlighting the need for fairer and more context-sensitive evaluation approaches in public education.

Policymakers should be aware/take into consideration that school size alone does not determine efficiency, and smaller schools may operate efficiently given their specific constraints and environments. In light of these results, uniform efficiency improvement strategies may not be appropriate. Instead, differentiated policy measures are recommended, tailored to the specific conditions and capacities of schools of different sizes. In general, the results of this paper can support evidence-based decision-making in education planning. By identifying schools with high efficiency and those with management improvement potential, local and national authorities can allocate resources more effectively, promote knowledge sharing between schools, and enhance the overall quality and efficiency of public education.

A key limitation of the analysis lies in the availability and comprehensiveness of the input and output data. Expanding the set of performance indicators – especially qualitative or curriculum-related dimensions – and applying mixed methods, such as stakeholder surveys or case studies, would enrich future analyses. Therefore, the MSEY should consider producing and publishing additional data that could serve as valuable inputs and outputs for assessing the efficiency of public primary education. Furthermore, data on the characteristics of best practice schools and other contextual factors influencing performance would support a more in-depth interpretation of efficiency results. Relevant variables could include tracking average grades in the first year of secondary school, as a more direct and meaningful measure of long-term educational outcomes, pupil participation and success in academic competitions, the number and composition of school staff (e.g., teachers, teaching assistants, cleaners, cooks, janitors), teacher qualifications beyond the basic requirements, principal tenure, and the extent of extracurricular or curriculum-enriching activities. Additionally, information on schools' cultural and public engagement could offer further insights into their broader contributions to pupils' educational experiences. To complement quantitative data, surveys or interviews with school staff and community members could provide valuable perspectives on how efficiency in primary education is defined and perceived. For example, Burušić et al. [8] used survey data to investigate the determinants of school efficiency at the primary level. Regular implementation of such surveys would support the development of high-quality datasets for future research. Finally, to strengthen empirical findings, conducting case studies on selected schools could help uncover the underlying reasons for differences in efficiency scores, offering a more nuanced understanding of the factors driving variation in school performance.

Future research could go beyond the assessment of PTE by examining the scale efficiency of public primary schools, which considers whether schools operate at an optimal size to maximise educational outcomes while minimising resource use. Moreover, incorporating complementary methodologies such as SFA could provide deeper insights – not only identifying efficient schools but also exploring how different inputs and external contextual factors influence efficiency outcomes. Adopting this broader analytical approach would support a more comprehensive understanding of the drivers of school performance and help pinpoint areas with the greatest potential for improvement.

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Customer impatience in the $GI/M(k)/1/N$ queue with working vacation: Economic and performance analysis

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Abstract. This study investigates a $GI/M(k)/1/N$ queueing system with Bernoulli feedback and multiple working vacation policy. The model incorporates customer impatience and accounts for the retention of reneged customers. By employing the supplementary variable technique and recursive methods, we derive the steady-state distributions of the customer count at both pre-arrival and arbitrary epochs. Various system performance measures are analyzed. In addition, an economic evaluation is carried out. Numerical illustrations are provided using the R software to confirm the analytical findings and highlight the influence of queueing parameters on performance indicators and system costs.

Keywords: The $GI/M(k)/1/N$ queue, impatient customers, working vacation, Bernoulli feedback, cost model

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1. Introduction

Queueing theory offers effective tools for analyzing and addressing congestion issues across numerous real-world and industrial domains. Applications span a broad spectrum, including emergency response systems, computing environments, customer service centers, online platforms, communication infrastructures, and various service facilities such as banks and governmental institutions [8, 11, 20, 17].

Queueing systems involving working vacations have been widely studied due to their operational importance in various sectors, such as manufacturing, service networks, transportation, telecommunications and computing systems. Unlike traditional vacation models, where the server becomes completely inactive, working vacation systems permit service continuation at a reduced rate during vacation periods. This approach enhances resource utilization efficiency and minimizes total system downtime. Given their practical significance, these systems have attracted considerable academic interest, beginning with the foundational work of Servi and Finn [19], followed by further developments by Jain and Agrawal [12], Baba [1, 2], Banik et al. [3], Bouchentouf et al. [5], among others.

Despite the advantages offered by these systems, customer impatience remains one of the major challenges affecting their performance. Prolonged waiting times may prompt customers to exhibit behaviors such as reneging—leaving the system before receiving service—or balking—deciding not to enter the system at all. Such behaviors can significantly reduce service efficiency and result in substantial revenue losses, making customer impatience a critical factor in queueing system design. In recent years, working vacation queues that incorporate customer impatience

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have been the subject of numerous analytical studies. For instance, Selvaraju and Goswami [18] discussed a $M/M/1$ queue by evaluating the impact of single versus multiple working vacation policies in the presence of both balking and reneging. Then, Laxmi and Jyothsna [15] investigated a $GI/M/1/N$ queue incorporating balking behavior and multiple working vacations. Building on this work, Goswami [10] undertook a systematic analysis of a $GI/M/1/N$ queueing model with multiple working vacations, emphasizing the joint effects of balking and reneging on performance metrics.

As mentioned earlier, customer impatience can result in the loss of potential clients and a decline in a firm's revenue. To mitigate this issue, businesses adopt various customer retention strategies with varying degrees of effectiveness. Retaining customers is particularly vital for organizations that frequently encounter impatience-related challenges. Companies employ different approaches, such as increasing service rates or introducing additional service channels, to encourage customers to remain in the system. For a comprehensive discussion on customer retention mechanisms, interested readers are referred to the works of Kumar and Sharma [14], Madheswari et al. [16], Yang and Wu [21], Bouchentouf et al. [4], among others. In real-life scenarios, customers may re-enter the system due to dissatisfaction with the initial service, as observed in supermarkets, hospitals, post offices, banks, and computer networks. The Bernoulli feedback mechanism models this behavior by describing situations in which customers, after receiving unsatisfactory service, either permanently exit the system or rejoin the queue with a certain probability. Additional insights can be found in [13, 6, 7, 9]. In the present paper, we perform a stationary analysis of a $GI/M(k)/1/N$ Bernoulli feedback queue model that incorporates multiple working vacations, state-dependent service rates, customer reneging due to impatience, and the retention of reneged customers. Our study investigates the complex interplay among these factors, which are common in practical service systems, yet rarely examined together in the queueing literature. This work makes a contribution in this regard. Using supplementary variable and recursive techniques, we derive steady-state probabilities at pre-arrival and arbitrary epochs. Furthermore, we present various system performance metrics and establish the economic model. Moreover, we carry out a numerical analysis to illustrate how queueing parameters influence key performance measures and economic characteristics.

The rest of this paper is structured as follows. Section 2 outlines the main assumptions and introduces the notation used throughout the model. Section 3 presents a detailed analytical study and derives the key performance metrics. Section 4 develops the cost evaluation framework. Numerical results and discussions are given in Section 5. Finally, Section 6 concludes the paper.

2. Description of the model

We consider a $GI/M(k)/1/N$ queueing model that incorporates Bernoulli feedback and customer impatience behaviors (balking and reneging). The service rates are state-dependent during normal busy periods and working vacations, which include multiple working vacations. The assumptions and notations introduced below will be used consistently throughout the paper.

- The inter-arrival times of successive arrivals are assumed to be independent and identically distributed (i.i.d.) random variables with cumulative distribution function $A(u)$, probability density function $a(u)$ for $u \geq 0$, Laplace-Stieltjes transform (LST) $A^*(s)$, and mean inter-arrival time $1/\lambda = -A^{*(1)}(0)$, where $h^{(1)}(0)$ denotes the first derivative of $h(s)$ evaluated at $s = 0$.

- The system's capacity is limited to N customers. When a customer arrives and finds k customers in the system, they join the queue with probability θ_k (where $\overline{\theta}_k = 1 - \theta_k$ is the balking probability). Specifically, $\theta_0 = 1$, $0 \leq \theta_{k+1} \leq \theta_k \leq 1$ for $1 \leq k \leq N-1$, and $\theta_N = 0$ to prevent joining when the buffer is full.

- Service is provided by a single server following a FIFO queue discipline. Service durations are exponentially distributed with parameters μ_k. When the server has no customers to serve and becomes idle, it proceeds to a working vacation period. The working vacation times follow an exponential distribution with mean of $1/\phi$. During a working vacation period, the server operates at a reduced service rate compared to regular periods, reflecting lower resource availability. The service durations during the working vacation period are also exponentially distributed with parameter ν_k, where $\nu_k < \mu_k$ for $1 \leq k \leq N$. During both the busy period and the working vacation period, the server can change its service rate when serving the same customer, depending on the system state. The average service rates are $\nu = \sum_{k=1}^{N} \nu_k/N$ for working vacations and $\mu = \sum_{k=1}^{N} \mu_k/N$ for regular busy periods.
- Upon entering the system, if a customer encounters an unavailable server—whether during normal busy periods or working vacations—they activate an impatience timer T, assumed to follow an exponential distribution with rate parameter ξ. If their service is not completed before the timer expires, they may abandon the queue with probability α and will never return to the system. Alternatively, with probability $\overline{\alpha} = 1 - \alpha$, they may be retained in the system through a retention mechanism. It is assumed that the impatience timers of all customers are independent of the number of customers waiting in the queue.
- After receiving unsatisfactory service—whether during the working vacation or normal periods—a customer may return to the system with probability $\overline{\beta}$ as a feedback customer for another regular service, or leave the system permanently with probability β, where $\overline{\beta} + \beta = 1$. No distinction is made between new arrivals and feedback arrivals.

All random variables involved in the system–namely, impatience timers, service times, working vacation periods and inter-arrival times–are considered to be mutually independent.

3. Analysis of the model

In this section, we study the system in steady-state using the supplementary variable technique. The state of the system at time t can be described by a continuous-time Markov process $\{(L(t), S(t), U(t)), t \geq 0\}$, with state space expressed as

$$\Omega = \{(0, 0, u) \cup (k, 0, u) \cup (k, 1, u) : 1 \leq k \leq N, u \geq 0\},$$

where:

- $L(t)$ denotes the number of customers in the system at time t
- $U(t)$ denotes the remaining inter-arrival time until the next customer arrival
- $S(t)$ denotes the state of the server at time t, defined as

$$S(t) = \begin{cases} 0, & \text{if the server is in a working vacation period,} \\ 1, & \text{if the server is in a normal busy period.} \end{cases}$$

Let us define the joint probabilities as

$$P_{k,0}(u,t)du = \mathbb{P}(L(t) = k, u \leq U(t) < u + du, S(t) = 0), u \geq 0, 0 \leq k \leq N,$$

$$P_{k,1}(u,t)du = \mathbb{P}(L(t) = k, u \leq U(t) < u + du, S(t) = 1), u \geq 0, 1 \leq k \leq N.$$

Then, in steady-state, we have

$$P_{k,0}(u) = \lim_{t\to\infty} P_{k,0}(u,t), \;\; 0 \leq k \leq N, \quad P_{k,1}(u) = \lim_{t\to\infty} P_{k,1}(u,t), \;\; 1 \leq k \leq N.$$

3.1. Governing equations

By employing the probabilistic arguments and using the remaining inter-arrival time as the supplementary variable, we observe the state of the system at two consecutive time epochs, t and $t+dt$. Taking $\lim\limits_{t\to\infty}$ and after some simplification, we have the following set of differential difference equations.

$$-P_{0,0}^{(1)}(u) = \beta\mu_1 P_{1,1}(u) + \beta\nu_1 P_{1,0}(u), \tag{1}$$

$$\begin{aligned} -P_{k,0}^{(1)}(u) &= -[\beta\nu_k + \phi + (k-1)\alpha\xi]P_{k,0}(u) + [\beta\nu_{k+1} + k\alpha\xi]P_{k+1,0}(u) \\ &\quad + a(u)[\theta_{k-1}P_{k-1,0}(0) + \overline{\theta}_k P_{k,0}(0)] \ ,1 \le k \le N-1, \end{aligned} \tag{2}$$

$$-P_{N,0}^{(1)}(u) = -[\beta\nu_N + \phi + (N-1)\alpha\xi]P_{N,0}(u) + a(u)[\theta_{N-1}P_{N-1,0}(0) + P_{N,0}(0)], \tag{3}$$

$$-P_{1,1}^{(1)}(u) = -\beta\mu_1 P_{1,1}(u) + \phi P_{1,0}(u) + (\beta\mu_2 + \alpha\xi)P_{2,1}(u) + a(u)\overline{\theta}_1 P_{1,1}(0), \tag{4}$$

$$\begin{aligned} -P_{k,1}^{(1)}(u) &= -[\beta\mu_k + (k-1)\alpha\xi]P_{k,1}(u) + [\beta\mu_{k+1} + k\alpha\xi]P_{k+1,1}(u) \\ &\quad + \phi P_{k,0}(u) + a(u)[\theta_{k-1}P_{k-1,1}(0) + \overline{\theta}_k P_{k,1}(0)], \ \ 2 \le k \le N-1, \end{aligned} \tag{5}$$

$$\begin{aligned} -P_{N,1}^{(1)}(u) &= -[\beta\mu_N + (N-1)\alpha\xi]P_{N,1}(u) + \phi P_{N,0}(u) \\ &\quad + a(u)[\theta_{N-1}P_{N-1,1}(0) + P_{N,1}(0)], \end{aligned} \tag{6}$$

where $P_{k,0}(0), 0 \le k \le N$ and $P_{k,1}(0), 1 \le k \le N$ are the respective rate probabilities with the remaining inter-arrival time equal to zero denoting that an arrival is about to occur. Then, we introduce the following Laplace-Stieltjes transforms of the steady-state probabilities as

$$P_{k,0}^*(s) = \int_0^\infty e^{-su}P_{k,0}(u)du, \ \ 0 \le k \le N, \ P_{k,1}^*(s) = \int_0^\infty e^{-su}P_{k,1}(u)du, \ \ 1 \le k \le N.$$

Let $P_{k,0} = P_{k,0}^*(0), 0 \le k \le N$ and $P_{k,1} = P_{k,1}^*(0), 1 \le k \le N$, be the steady-state probabilities of k customers in the system when the server is in state $j = 0, 1$ at an arbitrary epoch.
Let $P_{k,0}^-, 0 \le k \le N$, and $P_{k,1}^-, 1 \le k \le N$, denote the steady-state probabilities at a pre-arrival epoch, that is, an arrival sees k customers in the system and the server is in state $j = 0, 1$, at the arrival epoch.

Theorem 1. The steady-state probabilities at an arbitrary epoch in terms of the steady-state probabilities at a pre-arrival epoch are respectively expressed as follows:

$$P_{N,0} = \frac{\lambda}{\eta_N}\theta_{N-1}P_{N-1,0}^-,$$

$$P_{k,0} = \frac{\eta_{k+1} - \phi}{\eta_k}P_{k+1,0} + \frac{\lambda}{\eta_k}\left(\theta_{k-1}P_{k-1,0}^- - \theta_k P_{k,0}^-\right), k = N-1, \ldots, 1,$$

$$P_{N,1} = \frac{\phi}{\gamma_N}P_{N,0} + \frac{\lambda}{\gamma_N}\theta_{N-1}P_{N-1,1}^-,$$

$$P_{k,1} = \frac{\gamma_{k+1}}{\gamma_k}P_{k+1,1} + \frac{\phi}{\gamma_k}P_{k,0} + \frac{\lambda}{\gamma_k}\left(\theta_{k-1}P_{k-1,1}^- - \theta_k P_{k,1}^-\right), k = N-1, \ldots, 2,$$

$$P_{1,1} = \frac{\gamma_2}{\gamma_1}P_{2,1} + \frac{\phi}{\gamma_1}P_{1,0} - \frac{\lambda}{\gamma_1}\theta_1 P_{1,1}^-,$$

where $P_{0,0}$ can be computed by using the normalization condition, that is,

$$P_{0,0} = 1 - \left(\sum_{k=1}^{N} P_{k,0} + \sum_{k=1}^{N} P_{k,1} \right),$$

with $\eta_k = \beta\nu_k + \phi + (k-1)\alpha\xi$ and,$\gamma_k = \beta\mu_k + (k-1)\alpha\xi$, for $1 \leq k \leq N$.

Proof. The steady-state probabilities at arbitrary epochs are obtained by following the same analytical approach adopted in [3, 15, 10], which is based on the supplementary variable technique combined with recursive methods. □

3.2. Measures of performance

In this subsection, we derive and outline several important performance metrics for the proposed model, based on the steady-state probabilities at arbitrary epochs as established in the preceding subsection. These performance measures are as follows:

- The probabilities that the server is idle during a working vacation period (Π_I), busy during a working vacation period (Π_W), and busy during a normal busy period (Π_B) are given by:

$$\Pi_I = P_{0,0}, \quad \Pi_W = \sum_{k=1}^{N} P_{k,0}, \quad \Pi_B = \sum_{k=1}^{N} P_{k,1}.$$

- The average number of customers in the system (L_s) and the average number of customers in the queue (L_q) are given by:

$$L_s = \sum_{k=1}^{N} k(P_{k,0} + P_{k,1}), \quad L_q = \sum_{k=2}^{N} (k-1)(P_{k,0} + P_{k,1}).$$

- The expressions for the average balking rate (B_r) and number of customers served (S_c) are as follows:

$$B_r = \sum_{k=1}^{N} \lambda\overline{\theta}_k(P_{k,0} + P_{k,1}), \quad S_c = \beta\sum_{k=1}^{N} \nu_k P_{k,0} + \beta\sum_{k=1}^{N} \mu_k P_{k,1}.$$

- The expressions for the average reneging rate (R_r) and retention rate (R_e) are as follows:

$$R_r = \sum_{k=1}^{N} (k-1)\alpha\xi(P_{k,0} + P_{k,1}), \quad R_e = \sum_{k=1}^{N} (k-1)\overline{\alpha}\xi(P_{k,0} + P_{k,1}).$$

Remark. Under the assumption of state-independent service rates ($\mu_k = \mu$ and $\nu_k = \nu$ for all $k = 1, \ldots, N$), with $\beta = 1$ (implying no feedback customers) and $\alpha = 1$ (indicating no retention of reneged customers), our model coincides with the queueing system examined by Goswami [10]. Furthermore, if we additionally assume that customers are patient ($\xi = 0$), our model aligns with the framework introduced by Laxmi and Jyothsna [15]. Finally, by further imposing the absence of balking behavior ($\theta_k = 1$ for all $0 \leq k \leq N-1$), our model simplifies to the system studied by Banik et al. [3].

4. Economic model

To construct the economic model, we consider the following cost (in unit) elements associated with different events:

- Costs incurred based on the server's operational state and waiting customers:

- C_1: Cost when the server is idle during a working vacation period.

- C_2: Cost when the server is busy during a working vacation period.

- C_3: Cost when the server is busy during a normal busy period.

- C_4: Cost when customers join the queue and wait for service.

- Costs associated with impatient customers:

- C_5: Cost when a customer balks. - C_6: Cost when a customer reneges.

- C_7: Cost when a reneging customer is retained.

- Service-related costs:

- C_8: Cost per service. - C_9: Cost of serving a feedback customer.

Let R represent the revenue earned per serviced customer. The economic measures of the system are defined as follows:

- The total expected cost incurred by the system per unit time (T_C) is computed as follows:

$$T_C = C_1\Pi_I + C_2\Pi_W + C_3\Pi_B + C_4L_q + C_5B_r + C_6R_r + C_7R_e + (\mu + \nu)(C_8 + \overline{\beta}C_9).$$

- The total expected revenue generated by the system per unit time (T_R) and the corresponding total expected profit (T_P) are computed as follows:

$$T_R = R \times S_c, \qquad T_P = T_R - T_C.$$

5. Numerical illustrations and discussion

This section presents numerical examples for the state-dependent service rates and multiple working vacations in a $GI/M(k)/1/N$ queue. Various inter-arrival time distributions, including deterministic (D) and Erlang-3 (E_3), are considered. The outcomes, illustrated through graphs and tables, reveal the influence of changing system parameters on key performance metrics such as total expected cost, total expected revenue, and total expected profit. To achieve this, we developed an R program, authored by us, to demonstrate the practical applicability of the formulas derived in the previous sections. For this numerical study, we selected arbitrary values for different system parameters and costs. The cost elements used in the study are $C_1 = 5$, $C_2 = 7$, $C_3 = 10$, $C_4 = 10$, $C_5 = 4$, $C_6 = 4$, $C_7 = 8$, $C_8 = 10$, $C_9 = 5$, and $R = 100$. The following cases are considered:

- $\theta_k = 1 - \dfrac{k}{N}$, $\mu_k = 1.2k$, $\nu_k = 0.8k$, $\phi = 4.2$, $\overline{\beta} = 0.3$, $\overline{\alpha} = 0.4$, and $N = 8$. This case is presented in Table 1 and Figure 1.

- $\theta_k = e^{-0.1k}$, $\mu_k = 1.4k$, $\nu_k = 0.7k$, $\phi = 5.0$, $\lambda = 3.0$, $\xi = 0.8$, and $N = 9$. This case is presented in Figure 2 and Table 2.

		With balking			Without balking		
		$\lambda = 2.6$	$\lambda = 2.9$	$\lambda = 3.2$	$\lambda = 2.6$	$\lambda = 2.9$	$\lambda = 3.2$
$\xi = 0.6$	Π_I	0.186887	0.162467	0.142026	0.133298	0.107545	0.086755
	Π_W	0.105808	0.102906	0.099548	0.075771	0.068406	0.061075
	Π_B	0.707305	0.734628	0.758426	0.790931	0.824049	0.852169
	L_q	0.920353	1.049017	1.175376	1.418935	1.654920	1.892920
	B_r	0.563376	0.683875	0.813340	0.005055	0.010301	0.019089
	R_r	0.331327	0.377646	0.423135	0.510817	0.595771	0.681451
	R_e	0.220885	0.251764	0.282090	0.340545	0.397181	0.454301
	T_C	126.797565	129.129316	131.447604	131.583395	134.907988	138.248770
$\xi = 0.9$	Π_I	0.203947	0.179824	0.159471	0.152830	0.126448	0.104680
	Π_W	0.115384	0.113815	0.111691	0.086810	0.080370	0.073639
	Π_B	0.680669	0.706361	0.728838	0.760360	0.793182	0.821681
	L_q	0.820113	0.935488	1.049293	1.228458	1.432768	1.640300
	B_r	0.525254	0.636428	0.755929	0.002580	0.005380	0.010211
	R_r	0.442861	0.505164	0.566618	0.663367	0.773695	0.885762
	R_e	0.295241	0.336776	0.377745	0.442245	0.515797	0.590508
	T_C	126.569633	128.874890	131.172654	130.961746	134.197009	137.466633
$\xi = 1.2$	Π_I	0.218805	0.195175	0.175147	0.170192	0.143644	0.121390
	Π_W	0.123704	0.123443	0.122579	0.096603	0.091233	0.085331
	Π_B	0.657492	0.681382	0.702273	0.733204	0.765123	0.793279
	L_q	0.739351	0.843670	0.946880	1.081576	1.260746	1.443499
	B_r	0.494177	0.597580	0.708693	0.001392	0.002957	0.005717
	R_r	0.532333	0.607443	0.681754	0.778735	0.907737	1.039320
	R_e	0.354888	0.404962	0.454502	0.519157	0.605158	0.692880
	T_C	126.373521	128.650281	130.923132	130.448754	133.599580	136.795234

Table 1: *The impact of ξ and λ on system characteristics in the $D/M(k)/1/N$ queue model.*

Table 1 and Figure 1 illustrate the influence of the arrival rate λ and the reneging rate ξ on several key performance measures, as well as on the total expected cost, revenue, and profit, under the assumption of a deterministic inter-arrival time distribution.

- When the reneging rate ξ is fixed, an increase in the arrival rate λ results in higher values of the average number of customers in the system L_s and in the queue L_q, the probability that the server is in a normal busy state Π_B, the average number of customers served S_c, and the average rates of balking B_r, retention R_r, and reneging R_e. Conversely, the probabilities that the server is idle Π_I or working during a vacation period Π_W decrease. This is because a higher arrival rate increases the system congestion, which leads to greater impatience among customers and consequently a rise in reneging. Implementing customer retention strategies may help retain more customers in the system, thereby improving the average retention rate. Moreover, as L_s increases with λ, the average balking rate also rises. In addition, a higher λ increases the number of served customers, which in turn leads to higher values of total expected cost T_C, total expected revenue T_R, and total expected profit T_P.

- For a fixed arrival rate λ, increasing the reneging rate ξ causes decreases in B_r, L_q, L_s, S_c, and Π_B. Meanwhile, the average reneging and retention rates R_r and R_e, along with the probabilities Π_I and Π_W, increase. This behavior is due to the fact that a higher reneging rate reduces the average number of customers in the system and shortens waiting times, which leads to fewer customers choosing to wait (balking), and consequently lowers the average balking rate. However, the increase in customer loss caused by impatience reduces the total expected revenue T_R and total expected cost T_C, which ultimately results in a lower total expected profit T_P. The results conclusively demonstrate that customer

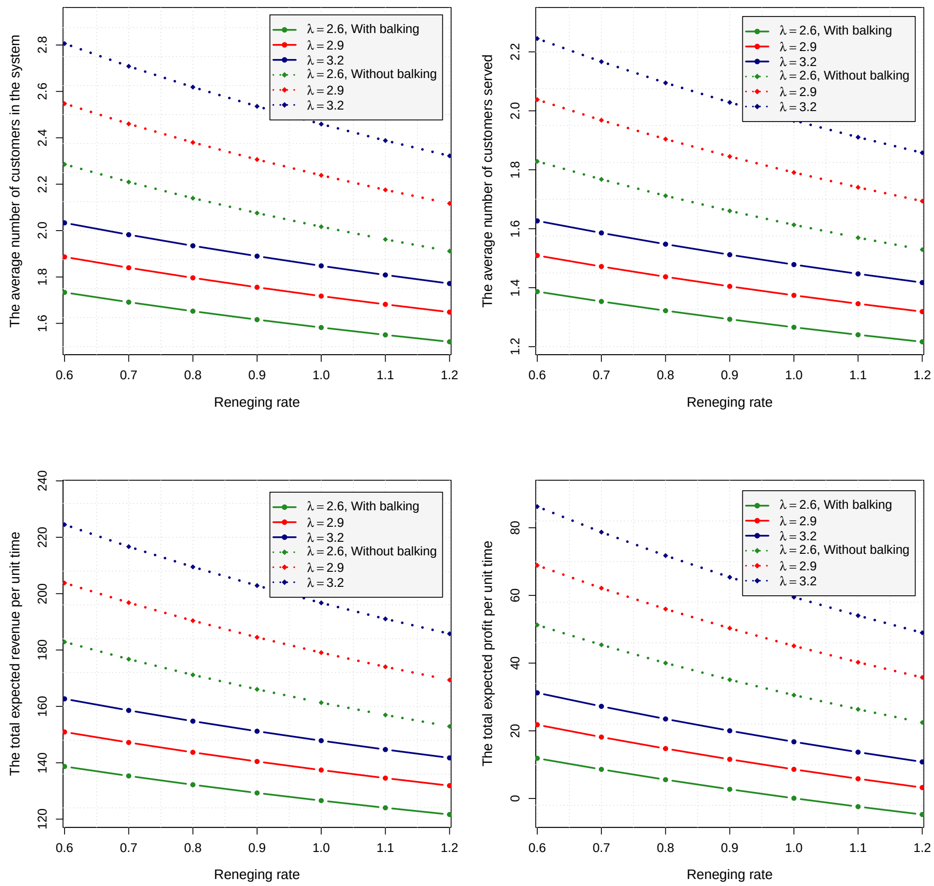

Figure 1: T*he impact of* ξ *and* λ *on* L_s, S_c, T_R *and* T_P.

impatience significantly degrades the system's economic performance metrics, confirming its substantial adverse effect on operational profitability.

Figure 2 and Table 2 illustrate the influence of the feedback probability $\overline{\beta}$ and the retention probability of reneged customers $\overline{\alpha}$ on various system performance indicators, as well as on the total expected cost, revenue, and profit, assuming that the inter-arrival times follow an Erlang-3 distribution. The main observations are as follows:

- When $\overline{\beta}$ is held constant, an increase in $\overline{\alpha}$ leads to growth in the average number of customers present in the system L_s and in the queue L_q, the average balking rate B_r, and the average retention rate R_e. In contrast, the average reneging rate R_r decreases, which aligns with expectations. Consequently, the probability that the server is operating in a regular busy period Π_B increases, while the probabilities of the server being idle Π_I or working during a vacation period Π_W decrease. These changes also result in an increase in the average number of customers served S_c. As $\overline{\alpha}$ increases, the total expected cost T_C, total expected revenue T_R, and total expected profit T_P all rise. Thus, a higher retention probability positively contributes to both operational efficiency and profitability.

- When $\overline{\alpha}$ is fixed, raising the feedback probability $\overline{\beta}$ results in increases in L_s, L_q, B_r, R_r, R_e, and S_c, as would be intuitively anticipated. This causes a drop in the probabilities

		With balking			Without balking		
		$\overline{\alpha} = 0.3$	$\overline{\alpha} = 0.5$	$\overline{\alpha} = 0.7$	$\overline{\alpha} = 0.3$	$\overline{\alpha} = 0.5$	$\overline{\alpha} = 0.7$
$\overline{\beta} = 0.3$	Π_I	0.186502	0.172795	0.157503	0.149592	0.134941	0.118978
	Π_W	0.104780	0.097120	0.088562	0.084230	0.076013	0.067049
	Π_B	0.708718	0.730085	0.753935	0.766178	0.789046	0.813974
	L_q	0.948604	1.043480	1.158561	1.244632	1.380842	1.545851
	B_r	0.471158	0.500069	0.535518	0.046996	0.065215	0.092240
	R_r	0.531218	0.417392	0.278055	0.696994	0.552337	0.371004
	R_e	0.227665	0.417392	0.648794	0.298712	0.552337	0.865677
	T_C	108.595008	110.813442	113.502055	111.336317	114.319574	117.965873
$\overline{\beta} = 0.5$	Π_I	0.098580	0.085862	0.072150	0.072766	0.060514	0.047900
	Π_W	0.056439	0.049173	0.041333	0.041727	0.034712	0.027485
	Π_B	0.844980	0.864964	0.886517	0.885506	0.904774	0.924616
	L_q	1.306120	1.461307	1.654829	1.707834	1.920901	2.180379
	B_r	0.590257	0.637761	0.699348	0.106555	0.153162	0.224286
	R_r	0.731427	0.584523	0.397159	0.956387	0.768360	0.523291
	R_e	0.313469	0.584523	0.926704	0.409880	0.768360	1.221012
	T_C	122.068469	125.476554	129.738197	125.995139	130.510280	136.115247
$\overline{\beta} = 0.7$	Π_I	0.034778	0.026437	0.018263	0.022303	0.015592	0.009653
	Π_W	0.020289	0.015426	0.010659	0.013024	0.009107	0.005639
	Π_B	0.944933	0.958137	0.971078	0.964673	0.975300	0.984707
	L_q	1.829330	2.088379	2.418199	2.355374	2.674136	3.050476
	B_r	0.757664	0.845329	0.968873	0.254126	0.375068	0.559279
	R_r	1.024425	0.835352	0.580368	1.319009	1.069654	0.732114
	R_e	0.439039	0.835352	1.354191	0.565290	1.069654	1.708267
	T_C	137.924211	143.335867	150.314194	143.443008	150.197202	158.496285

Table 2: *The impact of $\overline{\beta}$ and $\overline{\alpha}$ on system characteristics in the $E_3/M(k)/1/N$ queue model.*

Π_I and Π_W, and a corresponding rise in Π_B. Furthermore, increases in $\overline{\beta}$ translate into higher values of T_C, T_R, and T_P. These observations are in full agreement with real-world behavior, indicating that higher feedback probabilities strengthen the economic performance of the system.

Analyzing the data presented in Tables 1–2 and Figures 1–2, it is evident that the performance indicators Π_B, L_s, L_q, R_r, R_e, S_c, as well as T_C, T_R, and T_P attain higher values in the absence of balking compared to the scenario with balking. In contrast, the metrics Π_I, Π_W, and B_r exhibit increased values when balking behavior is included. These findings are fully consistent with the expected theoretical behavior of the system.

6. Conclusions and future work

This article investigates a non-Markovian queueing system with a limited buffer, where service rates vary depending on the system state during both normal busy and working vacation periods. The model incorporates the behavior of impatient customers as well as a Bernoulli feedback, all under a multiple working vacation policy. The developed framework is applicable to a broad range of practical environments, including industries facing congestion challenges, such as customer service centers, production lines, and communication infrastructures. Steady-state equations and system size probabilities at pre-arrival and arbitrary epochs are derived using the supplementary variable technique and recursive methods. Key performance measures are obtained, and an economic model is established. Numerical experiments evaluate the sensitivity of the system's performance metrics and economic indicators to variations in key parameters. The model can be further extended to accommodate multiple servers, as well as

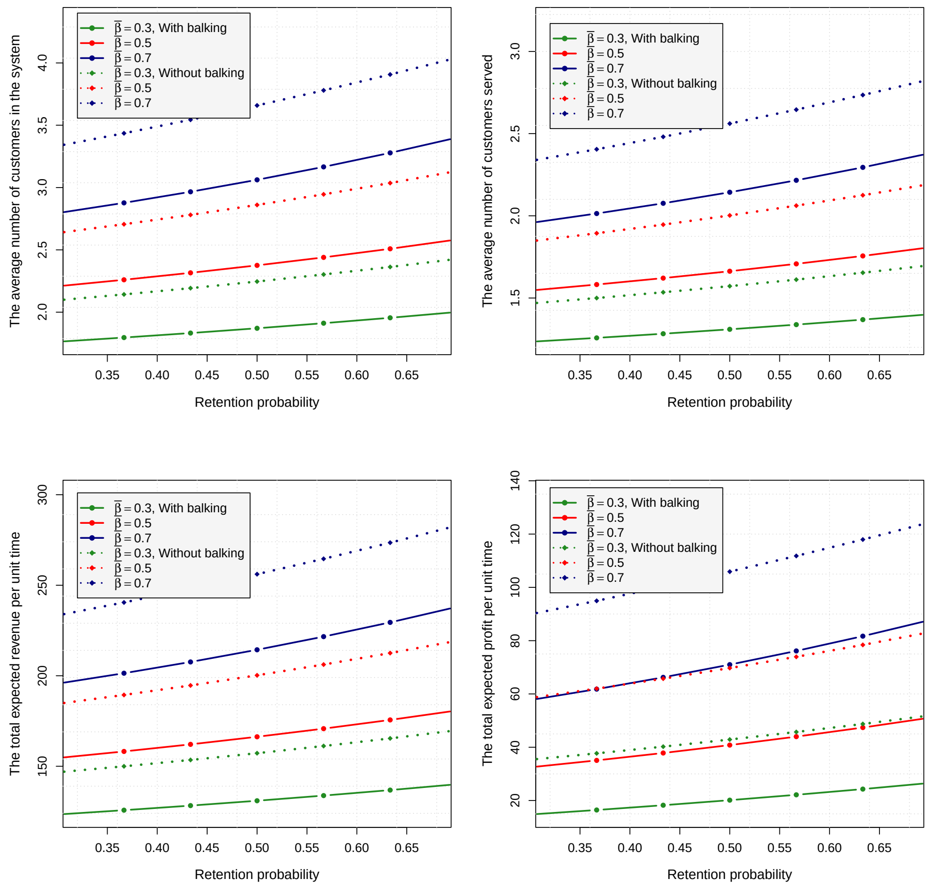

Figure 2: The impact of $\overline{\beta}$ and $\overline{\alpha}$ on L_s, S_c, T_R and T_P.

server breakdowns and repairs.

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Solution to the single parametric linear programming problems via simplex-based algorithms: handling the uncertainties in costs, left or right-hand sides

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Abstract. Parametric programming is one of the notable approaches to expressing the uncertainties encountered in real life. Many studies express the parameters of the objective function and right-hand side parametrically, but only a few include the parametric coefficient matrix of the constraints. This paper examines the feasibility and optimality conditions of the simplex table and proposes a simplex-based algorithm (dual-simplex, generalized-simplex, or primal-simplex). In the solution process, each case is considered independently through the mathematical analysis of simplex multipliers. Distinct numerical examples illustrate each case to demonstrate the algorithm's implementation.

Keywords: dual-simplex method, generalized-simplex method, parametric linear programming problem, primal-simplex method

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1. Introduction

The essential methods used to model uncertainty in the literature include deterministic, probabilistic, stochastic approaches, fuzzy mathematics, and interval arithmetic. The deterministic approaches contain mainly robust and parametric programming modeling. The parametric programming problem involves modeling uncertainties using parameters. It can be classified according to the location of the parameter (objective function, left-hand side (LHS), and right-hand side (RHS)) as well as the number of parameters (single or multi). Despite significant theoretical advancements in these classifications, it is observed that a limited number of studies have been conducted where the single parameter is on the LHS of the constraints in a Linear Programming (LP) problem.

A widely used algorithm in LP is a known simplex method that finds the optimal solution maximizing or minimizing a linear objective function, subject to linear (in)equality constraints. The method starts at a vertex of the feasible region and then moves from vertex to vertex along the edges of the feasible region. The method persists until it reaches the optimal vertex, where it either maximizes or minimizes the objective function. This method efficiently solves large-scale LP problems and is fundamental in operations research for problems such as resource allocation and production planning.

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Sensitivity analysis shows how parameter changes affect the optimal solution and examines how changes in the parameters affect the optimal solution. Ranges on the objective function coefficients and RHS values can describe the sensitivity of an optimal solution. These ranges were valid for one objective-function coefficient or RHS value change, while the remaining problem data are fixed. For instance, shadow prices determine changes in the objective function. Shadow prices, also known as dual prices, indicate how much the objective function would change if there is a one-unit increase in a constraint's RHS, assuming all other factors remain constant [10]. Therefore, a positive shadow price indicates that an increase in the resource will enhance the objective function and help prioritize resources with a higher marginal benefit. In contrast, a zero shadow price indicates a slight increase in the resource does not impact the objective. Sensitivity analysis helps determine the stability of the optimal solution and provides insight into which constraints or variables are most critical.

Khalilpour [6] presented a detailed chronological literature review on parametric optimization introduced in 1953. Here is a summary of recent literature in this field: Zuidwijk [17] provided the characterization of the objective function in terms of the parameter by applying realization theory to parametric sensitivity analysis. Ferris et al. [4] reviewed the sensitivity analysis for the parametric optimization of the objective function and RHS. In the study of Jongen et al. [5], one-parametric optimization problems were discussed, considering the local structure of the semi-infinite singularities. The parametric uncertainties in the objective function and RHS were looked into by Dua and Pistikopoulos [3] using a primal-simplex step to move from one basis to its neighbor. Using some classic results from matrix algebra, Khalilpour and Karimi [6] developed a two-stage iterative method to solve parametric optimization problems with a single parameter on the LHS. Charitopoulos [1] presented a method using Karush-Kuhn-Tucker conditions to solve a general Parametric LP (PLP) problem having uncertainties. In the study of Kolev and Skalna [7], an interval constraint satisfaction technique was proposed by obtaining the interval hull solution to a PLP problem. Yu and Monniaux [16] proposed a method using PLP for computations on polyhedral projection. In the study of Sivri et al. [11], an objective function with one parameter was converted into a linear structure by using a first-order Taylor series approximation. Mehanfar and Ghaffari [8] studied a uni-parametric linear program where an identical parameter perturbed the objective coefficients and the LHS and RHS of constraints linearly. They examined how the variation affects a specific optimal solution and how the optimal value function behaves within its domain.

In addition to theoretical research in the field of PLP, the use of such problems is also common in application areas. Wakili [13] dealt with an interval LP problem having a parameter in the objective function and provided a case study in the Coca-Cola company. In Chen [2], an LP technique for multidimensional analysis of preference methodology was utilized for addressing multiple criteria group decision-making problems based on Pythagorean fuzzy sets and by determining individual goodness of fit and poorness of fit. Widyan [15] presented a novel parametric method based on the simplex algorithm for decomposing the parametric space of the stochastic multicriteria optimization LP problem. The mathematical approach transformed the stochastic model with a random variable in the objective functions to a deterministic one, then scalarized the problem using the nonnegative weighted sum approach. Mousavi and Wu [9] suggested a framework based on parametric programming that would coordinate the work of the independent system operator and the distribution system operator. This would allow distributed energy resource aggregators to participate in the wholesale market as much as possible while ensuring that distribution grids run safely. Wakili [14] considered three types of bread with estimated profits and formulated a problem that was parameterized into PLP from the collected data. It was found that profit was made at different values of a parameter.

This study focuses on the optimization of LP problems with a single parameter. We propose a unified algorithm to obtain a solution for all cases where the parameter is on the objective function coefficients, the RHS, or the coefficient matrix of constraints. Using dual, generalized, or primal simplex algorithms, the proposed algorithm comes up with a parametric analytical solution based on the conditions of feasibility and optimality. Distinct numerical examples from the literature illustrate each case.

This paper is organized as follows: Section 2 gives the mathematical definition of the PLP problem and the proposed algorithm. After the implementation of numerical illustrations in Section 3, conclusions are interpreted in Section 4.

2. Proposed solution algorithm to PLP problem

This section initially outlines the fundamental framework of a PLP problem. After presenting different algorithms for each case where the parameter in the PLP problem lies in the objective function, LHS, or RHS, a unified algorithm for addressing each of these cases is proposed.

A general PLP problem with a single parameter can be defined as follows:

$$f_\lambda = min\left\{\mathbf{c}_\lambda\mathbf{x} : \mathbf{A}_\lambda\mathbf{x} \leq \mathbf{b}_\lambda, \mathbf{x} \geq \mathbf{0}\right\},$$

where λ is a parameter that reflects the uncertainty, A_λ is the coefficient matrix, $\mathbf{c}_\lambda$ and $\mathbf{b}_\lambda$ are the cost and RHS vectors in terms of λ, respectively.

PLP problems with a single parameter can be classified based on their location. If $\mathbf{c}_\lambda$, $\mathbf{A}_\lambda$, $\mathbf{b}_\lambda$ involve uncertainty, then the following PLP problems can be denoted by:

$$f_\lambda = min\left\{\mathbf{c}_\lambda\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\right\}, \tag{1}$$

$$f_\lambda = min\left\{\mathbf{c}\mathbf{x} : \mathbf{A}_\lambda\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\right\}, \tag{2}$$

or

$$f_\lambda = min\left\{\mathbf{c}\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}_\lambda, \mathbf{x} \geq \mathbf{0}\right\}. \tag{3}$$

While many approaches have been proposed in the literature for the solution of (1) and (3), there are only a few solution approaches for (2). Although we mainly focus on the solution of (2), this algorithm is designed to be effective in solving all types of one-parametric problems (1), (2), and (3).

We present the steps for each case before introducing the unified solution algorithm for PLP problems.

2.1. Solution steps for the LHS-PLP problem

The following steps are given for a PLP problem with the parameters on the LHS.

Step 1: Write the problem with m constraints in n variables in the standard form and construct the initial simplex table.

Step 2: Select the non-basic variables having the best costs in the direction of the objective function and obtain a $\lambda-$parametric simplex table by taking them into the base with $min\{m, n\}$ iteration(s). If the same costs appear in the objective function, break the ties arbitrarily.

Step 3: Let c_j be the coefficient of each variable in the objective function and j be the value found by multiplying each variable in a column with the corresponding coefficient of the basic solution variable in the objective function and summation of them. Identify λ values making shadow price $c_j - z_j$ for each non-basic variable, and $\mathbf{b}$ values equal to zero. Form a chart analyzing the signs of $c_j - z_j$ and $\mathbf{b}$ values in terms of the parameter λ to obtain possible optimal solutions.

Step 4: Make a joint sign chart having λ values in the columns and $\mathbf{b}$ and $c_j - z_j$ in the rows. To find the optimal basis for λ values, check the conditions for feasibility and optimality in each column of the joint sign chart and then apply one of the steps below:

Step 4a: If the RHS is negative and the optimality condition is satisfied, apply the dual-simplex algorithm.

Step 4b: If the RHS is negative and the optimality condition is not satisfied, apply the generalized-simplex algorithm.

Step 4c: If the RHS is positive and the optimality condition is not satisfied, apply the primal-simplex algorithm.

Step 5: State the final optimal basis for each interval.

2.2. Solution steps for the RHS-PLP problem

The solution to a PLP problem with parameters on the RHS is introduced through the following steps:

Step 1: Write the problem with m constraints in n variables in the standard form and construct the initial simplex table.

Step 2: Determine the entering variable by controlling the shadow prices, the $c_j - z_j$ values, taking into account the direction of the objective function.

Step 3: While determining the leaving vector, the minimum ratio test is applied by considering the row yielding all the parametric $\mathbf{b}$ values being non-negative.

Step 4: Save λ values, making $\mathbf{b}$ nonnegative in each iteration of the simplex method.

Step 5: Make a joint sign chart having λ values in the columns and $\mathbf{b}$ and final-basic variables in the rows. To find the optimal basis for λ values, check the conditions for feasibility and optimality in each column of the joint sign chart and then do one of the steps below:

Step 5a: If the RHS is negative and the optimality condition is satisfied, apply the dual-simplex algorithm.

Step 5b: If the RHS is negative and the optimality condition is not satisfied, apply the generalized-simplex algorithm.

Step 5c: If the RHS is positive and the optimality condition is not satisfied, apply the primal-simplex algorithm.

Step 6: State the final optimal basis for each interval.

2.3. Solution steps for the PLP problem with cost parameters

The solution to a PLP problem with parameters in the objective function is presented as follows:

Step 1: Write the problem with m constraints in n variables in the standard form and construct the initial simplex table.

Step 2: Considering the direction of the objective function (maximization or minimization), calculate the parametric shadow prices (if max, $c_j - z_j \geq 0$; if min, $c_j - z_j \leq 0$) for each vector that is not in the basis. Determine the entering variable by controlling the parametric $c_j - z_j$ values.

Step 3: Determine the λ values based on the parametric $c_j - z_j$ values concerning the direction of the objective function in each iteration of the simplex technique.

Step 4: Determine the leaving vector by applying the minimum ratio test.

Step 5: Make a joint sign chart having λ values in the columns, cost c_j of final basic variables, and shadow parametric prices $c_j - z_j$ variables in the rows. Considering the feasibility and optimality conditions in each column of the joint sign chart, apply one of the following steps to find the optimal basis relative to λ values:

Step 5a: If the RHS is negative and the optimality condition is satisfied, apply the dual-simplex algorithm.

Step 5b: If the RHS is negative and the optimality condition is not satisfied, apply the generalized-simplex algorithm.

Step 5c: If the RHS is positive and the optimality condition is not satisfied, apply the primal-simplex algorithm.

Step 6: State the final optimal basis for each interval.

2.4. Unified algorithm

The following steps can be expressed as a unified algorithm for solving all types of single PLP problems, such as (1), (2) or (3):

Step 1: Write the problem in the standard form and construct the initial simplex table.

Step 2: Select the non-basic variables having minimum costs in the objective function and obtain a $\lambda-$parametric simplex table by taking them into the base.

Step 3: Find the λ values that make $c_j - z_j$ equal to zero for each non-basic variable. Then, make a chart that shows how the sign of $c_j - z_j$ changes with λ to find possible optimal solutions.

Step 4: Considering the feasibility and optimality conditions in each column of the sign chart, apply one of the following to investigate the basis changes with λ values:

Step 4a: If the RHS is negative and the optimality condition is satisfied, apply the dual-simplex algorithm.

step 4b: If the RHS is negative and the optimality condition is not satisfied, apply the generalized-simplex algorithm.

Step 4c: If the RHS is positive and the optimality condition is not satisfied, apply the primal-simplex algorithm.

Step 5: State the final parametric solution for each interval.

The flowchart of the proposed algorithm is presented in Figure 1.

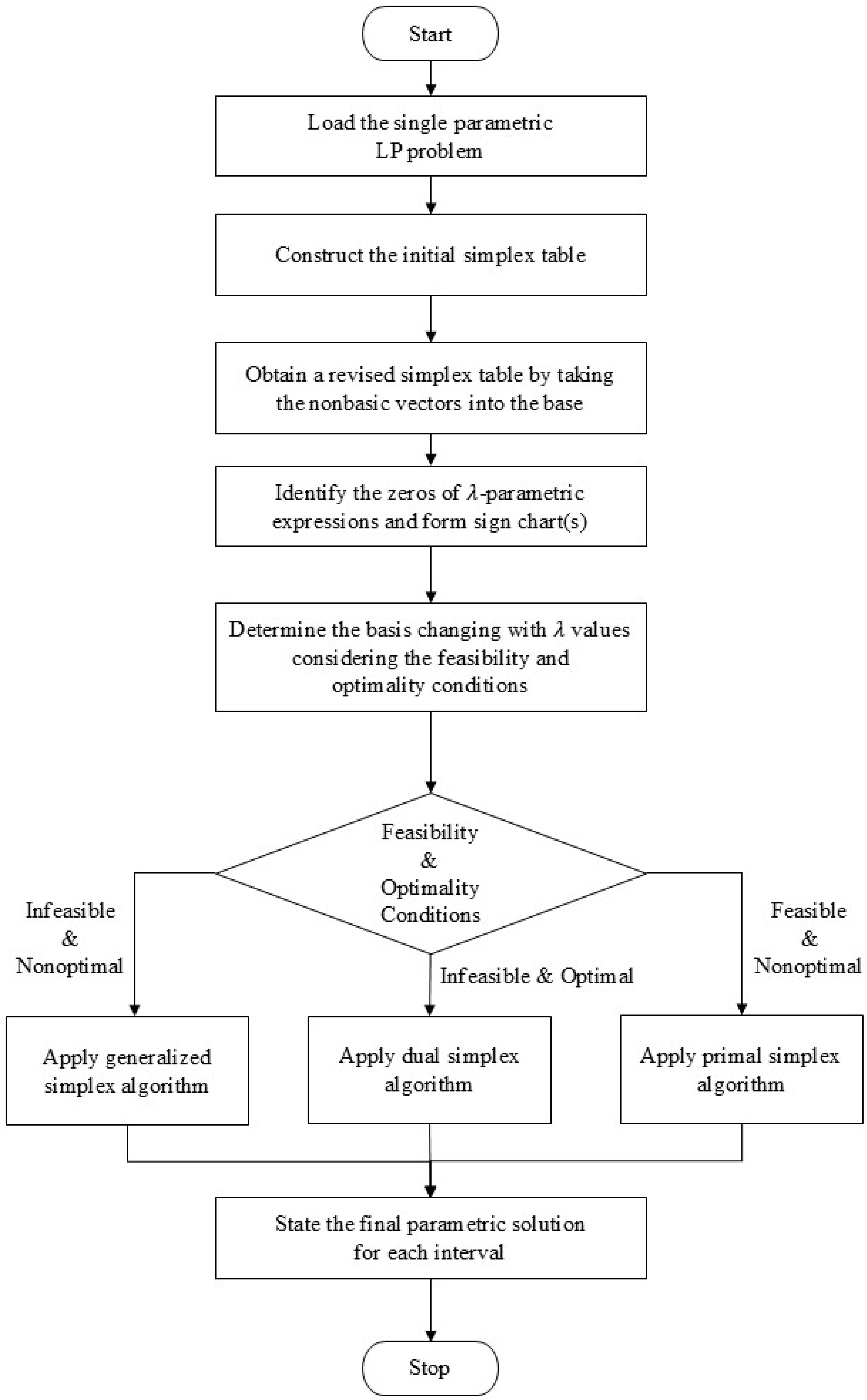

Figure 1: *Flowchart of the proposed algorithm.*

3. Numerical examples

This section specifically illustrates the solution algorithms for each case through numerical examples.

Example 1. *Consider the following PLP problem with a parameter in LHS, solved in the study [6]:*

$$min\ z = x_1 + 1.5x_2 + 3x_3 \tag{4a}$$

such that

$$x_1 + (1+2\lambda)x_2 + 2x_3 \geq 6,$$

$$(1+\lambda)x_1 + 2x_2 + x_3 \geq 10,$$

$$x_1, x_2 \geq 0.$$

First, the PLP problem is rewritten in the standard form and an initial simplex table is constructed as shown in Table 1.

x_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	**b**
x_6	1	$(1+2\lambda)$	2	-1	0	1	0	6
x_7	$(1+\lambda)$	2	1	0	-1	0	1	10

Table 1: *Initial simplex table*

There are two constraints in the PLP problem, and the variables x_1 and x_2 have the minimum costs in the objective function (4a), which means they should be minimized. Therefore, Table 1 is reformed by taking these variables into the basis using elementary row operations. Thus, Table 2 is constructed, where $\mathbf{b}^R$ is the RHS vector in the reformed simplex table.

x_B	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}^R$
x_1	1	0	$2+\dfrac{(1+2\lambda)^2}{1-3\lambda-2\lambda^2}$	$-1-\dfrac{1+3\lambda+2\lambda^2}{1-3\lambda-2\lambda^2}$	$\dfrac{1+2\lambda}{1-3\lambda-2\lambda^2}$	$6-\dfrac{4+2\lambda-12\lambda^2}{1-3\lambda-2\lambda^2}$
x_2	0	1	$\dfrac{-1-2\lambda}{1-3\lambda-2\lambda^2}$	$\dfrac{1+\lambda}{1-3\lambda-2\lambda^2}$	$\dfrac{-1}{1-3\lambda-2\lambda^2}$	$\dfrac{4-6\lambda}{1-3\lambda-2\lambda^2}$
$c_j - z_j$	0	0	$c_3 - z_3$	$c_4 - z_4$	$c_5 - z_5$	

Table 2: *Reformed simplex table*

In Table 2, $c_j - z_j, (j = 3, 4, 5)$ and **b** values depend on the parameter λ. To find the possible optimal solutions, all the zeros of λ-parametric expressions must be found. This creates a sign chart for each reduced cost and RHS value.

Equating $c_3 - z_3$ to zero,

$$3 - \left(2 + \frac{(1+2\lambda)^2}{1-3\lambda-2\lambda^2} + \frac{-1.5-3\lambda}{1-3\lambda-2\lambda^2}\right) = 0 \tag{5}$$

is obtained. The sign chart is constructed in Table 3 by considering zeros of (5), i.e., -1.78, -0.93, 0.27, and 0.28.

λ	-1.78	-0.93	0.27	0.28	
$c_3 - z_3$	$+$	$-$	$+$	$-$	$+$

Table 3: *Sign chart for $c_3 - z_3$*

For $c_4 - z_4$ and $c_5 - z_5$, the equations

$$0 - \left(-1 - \frac{1+3\lambda+2\lambda^2}{1-3\lambda-2\lambda^2} + \frac{1.5+1.5\lambda}{1-3\lambda-2\lambda^2}\right) = 0, \tag{6}$$

$$0 - \left(\frac{1+2\lambda}{1-3\lambda-2\lambda^2} - \frac{-1.5}{1-3\lambda-2\lambda^2}\right) = 0, \tag{7}$$

and the sign charts given in Table 4 and Table 5 are obtained, respectively.

λ	-1.78	0.28	0.33	
$c_4 - z_4$	$-$	$+$	$-$	$+$

Table 4: *Sign chart for $c_4 - z_4$*

λ	-1.78	0.25	0.28	
$c_4 - z_4$	$-$	$+$	$-$	$+$

Table 5: *Sign chart for $c_5 - z_5$*

Moreover, to find the feasible solutions, the RHS column of Table 2 should be positive, i.e., $\mathbf{b}^R \geq 0$. Therefore,

$$6 - \left(\frac{4+2\lambda-12\lambda^2}{1-3\lambda-2\lambda^2}\right) = 0,$$

and

$$\frac{4-6\lambda}{1-3\lambda-2\lambda^2} = 0$$

will be investigated. The sign chart for $\mathbf{b}_1^R$ and $\mathbf{b}_2^R$, the components of the vector $\mathbf{b}$ is presented in Table 6.

λ	-1.78	0.1	0.28	0.67	
$\mathbf{b}_1^R$	$-$	$+$	$-$	$+$	$+$
$\mathbf{b}_2^R$	$-$	$+$	$+$	$-$	$+$

Table 6: *Sign chart for $\boldsymbol{b}_1^R$ and $\boldsymbol{b}_2^R$*

Considering individual sign charts given in Table 3-5, the joint sign chart is presented in Table 7. By checking feasibility and optimality conditions for each interval in Table 7, generalized, dual, or primal simplex methods are applied to obtain possible optimal basis vectors. This analysis is summarized in Table 8. For example, for the interval $(-\infty, -1.78]$, the solution is found infeasible and nonoptimal from Table 8. Thus, the generalized-simplex method is applied, and the possible optimal basis vector is $\{x_3, x_2\}$.

λ	$c_3 - z_3$	$c_4 - z_4$	$c_5 - z_5$	$\mathbf{b}_1^R$	$\mathbf{b}_2^R$
$(-\infty, -1.78]$	$+$	$-$	$-$	$-$	$-$
$[-1.78, -0.93]$	$-$	$+$	$+$	$+$	$+$
$[-0.93, 0.1]$	$+$	$+$	$+$	$+$	$+$
$[0.1, 0.25]$	$+$	$+$	$+$	$-$	$+$
$[0.25, 0.27]$	$+$	$+$	$-$	$-$	$+$
$[0.27, 0.28]$	$-$	$+$	$-$	$-$	$+$
$[0.28, 0.33]$	$+$	$-$	$+$	$+$	$-$
$[0.33, 0.67]$	$+$	$+$	$+$	$+$	$-$
$[0.67, \infty)$	$+$	$+$	$+$	$+$	$+$

Table 7: *Joint sign chart for the reduced costs and RHS values*

Interval	Feasibility	Optimality	Simplex Method	Basis Vectors
$(-\infty, -1.78]$	Infeasible	Nonoptimal	Generalized	$\{x_3, x_2\}$
$[-1.78, -0.93]$	Feasible	Nonoptimal	Primal	$\{x_3, x_2\}$
$[-0.93, 0.1]$	Feasible	Optimal	Primal	$\{x_1, x_2\}$
$[0.1, 0.25]$	Infeasible	Nonptimal	Dual	$\{x_4, x_2\}$
$[0.25, 0.27]$	Infeasible	Nonoptimal	Generalized	$\{x_4, x_2\}$
$[0.27, 0.28]$	Infeasible	Nonoptimal	Generalized	$\{x_4, x_2\}$
$[0.28, 0.33]$	Infeasible	Nonoptimal	Generalized	$\{x_4, x_2\}$
$[0.33, 0.67]$	Infeasible	Nonoptimal	Dual	$\{x_1, x_4\}$
$[0.67, \infty)$	Feasible	Optimal	Primal	$\{x_1, x_2\}$

Table 8: *Possible optimal basis vectors based on the feasibility and optimality conditions*

Example 2. *Consider the following PLP problem having a parameter in RHS, solved in the study [1]:*

$$max\ z = 3x_1 + 2x_2 + 5x_3$$

such that $x_1 + 2x_2 + x_3 \leq 40 - \lambda,$

$$3x_1 + 2x_3 \leq 60 + 2\lambda,$$

$$x_1 + 4x_2 \leq 30 - 7\lambda,$$

$$x_1, x_2, x_3 \geq 0; \lambda \geq 0.$$

We first construct the initial simplex table, and then present the reformed simplex table as Table 9.

x_B	x_1	x_2	x_3	x_4	x_5	x_6	$\mathbf{b}^R$
x_2	-1/4	1	0	1/2	-1/4	0	$5 - \lambda$
x_3	3/2	0	1	0	1/2	0	$30 + \lambda$
x_6	2	0	0	-2	1	1	$10 - 3\lambda$
$c_j - z_j$	-4	0	0	-1	-2	0	

Table 9: *Reformed simplex table for the PLP problem including a RHS parameter*

The zeros of λ-parametric expressions are found, and the values making the RHS nonnegative are determined from Table 9. Since the parameter λ is defined as nonnegative, the sign chart is constructed as Table 10, and according to the sign chart, the optimal basis vectors can be presented in Table 11.

λ		10/3	30/7	5	40
$\mathbf{b}_1 = 40 - \lambda$	$+$	$+$	$+$	$+$	$-$
$\mathbf{b}_2 = 60 + 2\lambda$	$+$	$+$	$+$	$+$	$+$
$\mathbf{b}_3 = 30 - 7\lambda$	$+$	$+$	$-$	$-$	$-$
$\mathbf{b}_1^R = 5 - \lambda$	$+$	$+$	$+$	$-$	$-$
$\mathbf{b}_2^R = 30 + \lambda$	$+$	$+$	$+$	$+$	$+$
$\mathbf{b}_3^R = 10 - 3\lambda$	$+$	$-$	$-$	$-$	$-$

Table 10: *Sign chart for the RHS PLP problem*

Interval	Feasibility	Optimality	Simplex Method	Basis Vectors
$[0, 10/3]$	Feasible	Optimal	Primal	$x_2 = 5 - \lambda$ $x_3 = 30 + \lambda$
$[10/3, 30/7]$	Infeasible	Nonoptimal	Generalized	$x_2 = 15/2 - 7/4\lambda$ $x_3 = 30 + \lambda$
$[30/7, 5]$	Infeasible	Nonoptimal	Generalized	No feasible solution
$[5, 40]$	Infeasible	Nonoptimal	Generalized	No feasible solution
$[40, \infty)$	Infeasible	Nonoptimal	Generalized	No feasible solution

Table 11: *Possible optimal solutions for the RHS PLP problem*

Example 3. *Consider the PLP problem with a parameter in the objective function, taken from [12]:*

$$max\ z = (3 - 6\lambda)x_1 + (2 - 2\lambda)x_2 + (5 + 5\lambda)x_3 \tag{9a}$$

such that

$$x_1 + 2x_2 + x_3 \leq 40,$$

$$3x_1 + 2x_3 \leq 60,$$

$$x_1 + 4x_2 \leq 30,$$

$$x_1, x_2, x_3 \geq 0; \lambda \geq 0.$$

The initial simplex table is constructed, and at each step, the vectors maximizing the objective function are taken into the basis. The reformed simplex table is presented in Table 12.

x_B	x_1	x_2	x_3	x_4	x_5	x_6	$\mathbf{b}^R$
x_2	0	1	0	1/4	-1/8	1/8	25/4
x_3	0	0	1	3/2	-1/4	-3/4	45/2
x_1	1	0	0	-1	1/2	1/2	10/3
$c_j - z_j$	0	0	0	$-5 - 13\lambda$	4λ	$2 + 7\lambda$	

Table 12: *Reformed simplex table for PLP problem*

Since the parameter λ is defined as nonnegative, the only root of the expressions in λ found from the reduced cost row is $\lambda = 0$. Also, zeros of the expressions in λ found from the objective function (9a) are $\lambda = 0.5$ and $\lambda = 1$. The sign chart for the PLP problem is constructed as Table 13 and, according to the sign chart, the possible optimal solutions are presented in Table 14.

λ		0.5	1
$c_1 = 3 - 6\lambda$	$+$	$-$	$-$
$c_2 = 2 - 2\lambda$	$+$	$+$	$-$
$c_3 = 5 + 5\lambda$	$+$	$+$	$+$
$c_4 - z_4 = -5 - 13\lambda$	$-$	$-$	$-$
$c_5 - z_5 = 4\lambda$	$+$	$+$	$+$
$c_6 - z_6 = 2 + 7\lambda$	$+$	$+$	$+$

Table 13: *Sign chart for the PLP problem*

Interval	Feasibility	Optimality	Simplex Method	(x_1, x_2, x_3)	Objective Function
$[0, 0.5]$	Feasible	Nonoptimal	Primal	$(0, 5, 30)$	$z = 160 + 140\lambda$
$[0.5, 1]$	Feasible	Nonoptimal	Primal	$(0, 5, 30)$	$z = 160 + 140\lambda$
$[1, \infty)$	Feasible	Nonoptimal	Primal	$(0, 0, 30)$	$z = 150 + 150\lambda$

Table 14: *Possible optimal solutions for the PLP problem*

4. Conclusion

The proposed algorithm provides a systematic approach to PLP problems with a single parameter and generates a solution if the parameter is only in the objective function, only on the RHS, or only in the coefficient of constraints. To the best of our knowledge, the literature lacks a solution approach for all cases, indicating the algorithm's usefulness. Additionally, the study has made a significant contribution to the limited literature in this field, specifically addressing the PLP problem when a parameter falls on the lower bound of the constraints. We propose a traditional approach to real-life problems to mitigate the uncertainty arising from the parameter's location. We can present future work that adapts the proposed approach to multi-parameter PLP problems.

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On the identification of a distribution for the arrivals to a queueing system: an application from bank data

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Abstract. Queues are an integral part of life. The literature available concentrates on Poisson arrival and exponential service time. The present paper focuses on statistical inference on the parameters of queueing models. The work is data driven. The dataset considered is the queueing system in banks which is publicly available. The number of servers has been estimated for a steady state system. The data has been tested for normality. The goodness of fit test has been done for the arrival pattern, and it is found that the discrete analog of smallest extreme value distribution is a good fit to the data which is an alternative to the widely used Poisson distribution.

Keywords: arrival time, model identification, queue, service time, waiting time

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1. Introduction

Queues are formed when there is heavy demand for service. The characteristics of a queue are the arrival pattern, the service rate, the system capacity and the queue discipline. The widely used distribution for arrival pattern is the Poisson distribution, and for the service rate, it is the exponential distribution. The system capacity may be finite or infinite, and the queue discipline may be first come first served or based on priority. These characteristics play a vital role in the waiting time of the customers, and consequently the queue length. The customers who need service may be either animate or inanimate. Extensive work on queueing theory can be found in Bhat [3], Gross et al. [8] and the references therein.

The recent pandemic COVID 19 was a period of crisis for the entire world. The application of queueing theory was conspicuously seen in the health sector. This is not a surprise as its history races back to Wolff [16] where the author discusses the problems of statistical inference for birth and death models. Besides the health sector, the effect was seen in banking and economy too despite digitalization. Dehmi et al. [6], Sreelatha et al. [13] and Ramesh and Manoharan [11] are some of the recent works in this direction. Adaptive Neuro-Fuzzy Inference System (ANFIS) was used to compute and optimize the cost of a queueing system [6]. A high performing probabilistic model was proposed for systems with dynamic service facility [13]. Comparison of blocking mechanisms with capacity restrictions was carried out in a two-station tandem network queueing model [11].

Some of the works related to the banking sector are that by Afolalu et.al. [1], Al-Jumaily and Al-Jobori [2], Cowdrey et al. [5], Eze and Odunukwe [7], Joel and Augustine [9], Sy et al.

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[15] and Yifter et al. [17]. An overview of applications of queueing models is given in [1], while development of automatic queueing system based on different queueing disciplines was carried out by [2]. Also, Markovian queueing model was applied to analyze customer management, optimize the waiting times and, improve service quality ([5], [7], [9], [15]). Recently, assessment of quality of services was examined using modeling and simulation [17].

Sampling design is an important framework for any probability or stochastic modeling. This includes estimation and inference of the parameters of the model. The terminology used for the process is known as identification. Queueing model is a stochastic model and the literature available on it widely focuses on Poisson distribution for the arrival of customers. Since the present-day studies are evident based or data driven, we decided to focus on fitting a probability distribution to an empirical data for validating whether Poisson distribution is always a good fit to the arrival pattern. The data used are available in Bishop et al. [4] on the arrivals of customers to three banks in Nigeria.

The main contribution of the present work is

- Estimation of the number of servers required at steady state
- Fitting a model to the arrival pattern of customers to a bank and
- Testing for goodness of fit of the model

The organization of the paper is as follows. Section 2 discusses the means of obtaining the various performance measures of a queueing system. Section 3 gives a brief description of the data. The identification of the distribution to the arrival pattern is done using goodness of fit test. This is shown in section 4. Section 5 gives the results of the analysis. The conclusion is given in section 6.

2. Performance measures of a queueing system

The following measures need to be considered in any general queueing system.

Let $N_q(t)$ and $N_s(t)$ denote respectively, the number of customers in the queue and the number at service, at time t, $t \geq 0$. Then, the total number of customers in the system is given by $N(t) = N_q(t) + N_s(t)$, $t \geq 0$.

Let $p_n(t) = P(N(t) = n)$ be the probability that there are n customers in the system at time t, $t \geq 0$, and its steady state probability be $p_n = P(N = n)$. Then, the expected number of customers in the system at steady state is $L = E(N) = \sum_{n=0}^{\infty} p_n$ and that in the queue is $L_q = E(N_q) = \sum_{n=s+1}^{\infty} (n-s)p_n$. Here, s represents the number of servers. Thus, the expected number of customers in service in the steady state is $E(N_S) = L - L_q$.

Let T_q be the time a customer spends waiting in the queue prior to entering service and T be the total time a customer spends in the system. Then $T = T_q + S$, where S is the service time. Since T, T_q and S are random variables, the mean waiting time in the queue is $W_q = E(T_q)$ and that in the system is $W = E(T) = E(T_q) + E(S)$.

The relationship between waiting time and queue length was given by J. D. C. Little. The equations are $L = \alpha W$ and $L_q = \alpha W_q$. Here, α is the average arrival rate of customers.

If β is the average service rate, then $\rho = \alpha/s\beta$ is a measure of traffic congestion. This is also known as traffic intensity. The system is said to be in a steady state when $\rho < 1$, i.e., the arrival rate is less than the service rate. Another important terminology in queueing theory is the busy period. This is the time from when a customer enters an empty system until it next empties out again. The empirical expression for arrival rate may also be written as $\alpha = \frac{N_c}{T}$, where N_c is the number of customers arriving over a period $(0, T)$.

Theoretically, the expression for p_n may be derived from the differential-difference equations of the general birth-death model. The equations are as follows.

$$\frac{dp_n(t)}{dt} = -(\alpha_n + \beta_n)p_n(t) + \alpha_{n-1}p_{n-1}(t) + \beta_{n+1}p_{n+1}(t), \;\; n \geq 1 \tag{1}$$

and

$$\frac{dp_0(t)}{dt} = -\alpha_0 p_0(t) + \beta_1 p_1(t) \tag{2}$$

where, α_n and β_n are the transition rates of birth and death respectively and n is the population size. Here, $p_n(t)$ is the unconditional probability that the system is in state n at time t, $t \geq 0$. Equations (1) and (2) are forms of the Chapman-Kolmogorov forward equations. Further information on these system of differential-difference equations are provided in [3].

The steady state solutions for the equations (1) and (2) are

$$p_n = p_0 \prod_{i=1}^{n} \frac{\alpha_{i-1}}{\beta_i}, \;\; n \geq 1 \tag{3}$$

and

$$p_0 = \left[1 + \sum_{n=1}^{\infty} \prod_{i=1}^{n} \frac{\alpha_{i-1}}{\beta_i}\right]^{-1} \tag{4}$$

respectively. Here, p_n is the probability of having n individuals in the system and p_0 is the probability that the system is empty initially. More precisely, the limiting distribution of the state of the birth-and-death queueing model are $\{p_n,\, n = 0, 1, \cdots\}$ and that $\{p_n,\, n = 0, 1, \cdots\}$ are nonzero if and only if $1+\sum_{n=1}^{\infty} \prod_{i=1}^{n} \frac{\alpha_{i-1}}{\beta_i} < \infty$. Also, the normalizing condition $\sum_{n=1}^{\infty} p_n = 1$ holds.

If $\alpha_n = \alpha$ and $\beta_n = \beta$, the general birth-and-death model reduces to the simplest queueing model $M/M/s$, where M stands for Markovian. The arrival process Poisson and service process exponential are both Markovian. Thus, for a single server system, the probability of having no customer in the system is $p_0 = (1 - \frac{\alpha}{\beta})$ and the probability of having n customers in the system is $p_n = (1 - \frac{\alpha}{\beta}) \left(\frac{\alpha}{\beta}\right)^n$, $n \geq 1$. In this case, the traffic intensity will be $\rho = \frac{\alpha}{\beta}$, and system works in the steady state if $0 < \rho < 1$.

The details may be found in standard textbooks such as Bhat [3] and Gross et al. [8].

3. Data description

The dataset from Bishop et al. [4] consists of the number of customers arriving to three different banks of three different urban areas in Ogun State, Nigeria. The data give the number of customers arriving on twenty different days in four weeks. Let this be denoted as N. At each bank, the time range of a customer's arrival, the time his/her cheque/withdrawal booklet was collected, the time used to process it and the total time spent by the customers in the bank were recorded. Let these random variables be denoted respectively as W_q, S and W. Thus, using the terminologies of the queueing theory, W_q represents the waiting time of customers in queue, S the service time, and W the total time spent by the customer in the bank. The observations are made for seven hours each in all three banks.

The number of arrivals on twenty days to the three banks is given in Table 1. The corresponding bar chart is shown in Figure 1.

Day	1	2	3	4	5	6	7	8	9	10
Bank-I	880	720	1020	802	522	989	684	548	1021	789
Bank-II	1034	789	1002	910	931	748	924	872	764	890
Bank-III	767	930	921	878	790	876	923	910	1002	949
Day	11	12	13	14	15	16	17	18	19	20
Bank-I	1000	990	1001	1051	982	857	981	1057	899	996
Bank-II	971	685	724	873	605	1017	1009	891	948	901
Bank-III	934	1011	874	762	631	989	784	648	891	752

Table 1: *Number of arrivals to the three banks*

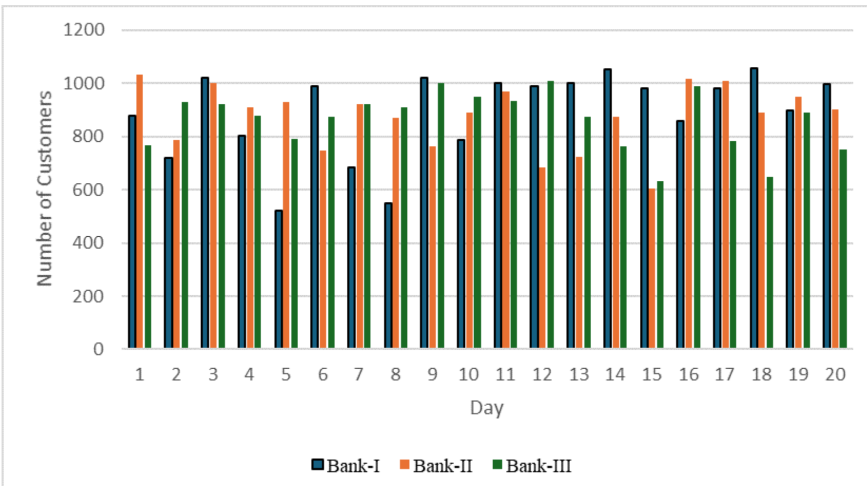

Figure 1: *Bar chart representation of the number of customers arriving to the three banks*

Table 2 gives the descriptive statistics of the data.

Bank	Statistic	Waiting Time in Queue (W_q) (in minutes)	Service Time (S) (in minutes)	Waiting Time in System (W) (in minutes)	Number of Arrivals per Day (N)
Bank-I	Average	12.65	15.1	27.75	889.45
	SD	6.64	5.46	9.07	162.71
Bank-II	Average	16.05	14.7	30.75	874.4
	SD	6.35	6.47	7.47	119.13
Bank-III	Average	11.85	11.15	23	861.1
	SD	3.94	3.51	5.77	109.49

Table 2: *Descriptive statistics*

Table 3 shows the arrival rate (α), service rate(β) and the number of servers required at each bank under the steady state.

Bank	Arrival Rate (α) (per hour=$N/7$)	Service Rate (β) (per hour=$60 diagup S$)	Number of Servers Required (s)
Bank-I	127.06	3.97	33
Bank-II	124.91	4.08	31
Bank-III	123.01	5.38	23

Table 3: *Arrival rate, service rate and number of servers at steady state*

In Bank-I, the average arrival is 127.06 per hour and the average service time is 3.97 per hour. Since, we are considering a steady state system, the condition $0 < \rho < 1$ should hold. Now, $\rho = \dfrac{\alpha}{\beta} = \dfrac{127.06}{3.97} = 32.005 > 1$.

Hence, if we set $s = 32$, and compute $\rho = \dfrac{\alpha}{s\beta} = \dfrac{127.06}{(32 \times 3.97)} = 1.00015 > 1$.

Since it is known that a queue will build up infinitely when $\rho \geq 1$ ([3], [8]), to bring the system under steady state, we increase the value of s and compute corresponding ρ until $\rho < 1$. When $s = 33$, $\rho = \dfrac{\alpha}{s\beta} = \dfrac{127.06}{(33 \times 3.97)} = 0.9698 < 1$, and when $s = 34$, $\rho = \dfrac{127.06}{(34 \times 3.97)} = 0.9413 < 1$.

It can be observed from the above calculations that, when the value of $s \geq 33$, the system is under steady state. The additional servers induce cost. Hence, if 33 servers are employed to attend to the large number of arrivals, the system will be maintained at steady state in Bank-I. The same explanation holds for Bank-II and Bank-III.

4. Goodness of fit test for arrival pattern

Since the main objective of this work is to find an appropriate distribution (which could also be an alternative distribution to Poisson) for the arrival pattern, the goodness of fit test was performed on the data. This was done using the trial version of SPC for Excel Software. Although arrival pattern is a discrete process, the software considers only the following continuous distributions. The probability density function (p. d. f.) and the corresponding log-likelihood function of the distributions tested for the arrivals are provided in Table 4.

The log-likelihood of the specified distribution is given for a sample size n with values $x_1, x_2, \cdots,$
x_n. In Table 4, μ. σ, α and θ refer to the location, scale, shape and threshold parameters respectively. More information related to these continuous distributions can be found in Johnson et al. [10].

The tests used were Shapiro-Wilk test (SW), Cramer-von Mises test (CVM) and Anderson-Darling test (AD). While SW test is used to test for the normality of the data, the CVM and AD tests are based on empirical distribution function (Stephens [14]). The hypothesis used in these goodness of fit tests is, H_0: The sample data follow the hypothesized distribution. If $x = x_1, x_2, \cdots, x_n$ denotes a random sample following a specified distribution, then the
SW test statistic is given by

$$W = \frac{\left(\sum_{i=1}^{n} a_i x_{(i)}\right)^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

where $x_{(i)}$ is the i^{th} order statistic.
CVM test statistic is given by

$$W^2 = \sum_{i=1}^{n} \left[F(X_i) - \frac{(2i-1)}{2n}\right]^2 + \frac{1}{12n}$$

and
AD test statistic is given by

$$A^2 = -n - \frac{\left[\sum_{i=1}^{n}(2i-1)\left\{ln(F(X_i)) + ln(1-F(X_{n+1-i}))\right\}\right]}{n}$$

where $F(.)$ is the cumulative distribution function of the specified distribution.

Distribution	p.d.f ($f(x)$)	Log-Likelihood
Smallest Extreme Value	$\frac{1}{\sigma}exp\left(\frac{x-\mu}{\sigma}\right)exp\left(-e^{\left(\frac{x-\mu}{\sigma}\right)}\right)$, $-\infty < x < \infty;\ \sigma > 0$	$\sum_{i=1}^{n}\left[\frac{x_i-\mu}{\sigma} - ln(\sigma) - exp(\frac{x_i-\mu}{\sigma})\right]$
Weibull	$\frac{\alpha}{\sigma}(\frac{x}{\sigma})^{(\alpha-1)}exp((-\frac{x}{\sigma})^{\alpha})$, $x \geq 0;\ \alpha,\ \sigma > 0$	$\sum_{i=1}^{n}[ln(\alpha) + (\alpha-1)ln(x_i) - \alpha ln(\sigma) - (\frac{x_i}{\sigma})^{\alpha}]$
Lognormal-Three Parameter	$\frac{1}{\sigma(x-\theta)\sqrt{2\pi}}exp\left(\frac{-(ln(x-\theta)-\mu)^2}{2\sigma^2}\right)$, $x > \theta;\ -\infty < \mu < \infty;\ \sigma > 0$	$-\sum_{i=1}^{n}\left[ln(\sigma\sqrt{2\pi}) + ln(x_i-\theta) + \frac{(ln(x_i-\theta)-\mu)^2}{2\sigma^2}\right]$
Normal	$\frac{1}{\sigma\sqrt{2\pi}}exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$, $-\infty < x < \infty;\ \sigma > 0$	$-\sum_{i=1}^{n}\left[ln(\sigma\sqrt{2\pi}) + \frac{(x_i-\mu)^2}{2\sigma^2}\right]$
Gamma	$\frac{x^{\alpha-1}}{\Gamma(\alpha)\sigma^{\alpha}}exp(\frac{-x}{\sigma}),\ \ x > 0;$ $\alpha,\ \sigma > 0$	$\sum_{i=1}^{n}\left[(\alpha-1)ln(x_i) - \frac{x_i}{\sigma} - ln(\Gamma(\alpha)) - \alpha ln(\sigma)\right]$
Loglogistic	$\frac{\alpha\sigma^{\alpha}x^{\alpha-1}}{(x^{\alpha}+\sigma^{\alpha})^2},\ x \geq 0;\ \alpha,\ \sigma > 0$	$\sum_{i=1}^{n}[ln(\alpha) + \alpha ln(\sigma) + (\alpha-1)ln(x_i) - 2ln(x_i^{\alpha}+\sigma^{\alpha})]$
Lognormal	$\frac{1}{\sigma x\sqrt{2\pi}}exp\left(\frac{-(ln(x)-\mu)^2}{2\sigma^2}\right)$, $x > 0;\ -\infty < \mu < \infty;\ \sigma > 0$	$-\sum_{i=1}^{n}\left[ln(\sigma\sqrt{2\pi}) + ln(x_i) + \frac{(ln(x_i)-\mu)^2}{2\sigma^2}\right]$
Largest Extreme Value	$\frac{1}{\sigma}exp\left(-\frac{(x-\mu)}{\sigma}\right)exp\left(-e^{-\left(\frac{x-\mu}{\sigma}\right)}\right)$, $-\infty < x < \infty;\ \sigma > 0$	$\sum_{i=1}^{n}\left[exp(\frac{x_i-\mu}{\sigma}) - ln(\sigma) - \frac{x_i-\mu}{\sigma}\right]$
Exponential-Two Parameter	$\frac{1}{\sigma}exp(\frac{-(x-\theta)}{\sigma}),\ \ x \geq \theta;\ \sigma > 0$	$-\sum_{i=1}^{n}\left[ln(\sigma) + \frac{x_i-\theta}{\sigma}\right]$
Exponential	$\frac{1}{\sigma}exp(\frac{-x}{\sigma}),\ \ x \geq 0;\ \sigma > 0$	$-\sum_{i=1}^{n}\left[ln(\sigma) + \frac{x_i}{\sigma}\right]$

Table 4: *Probability distributions and log-likelihood functions*

The functions *shapiro.test()* and *cvm.test()* in R [12] were used for SW and CVM tests.

The AD test was performed using SPC for Excel Software. This software reports the test statistic values, the p-values and the Akaike Information Criteria (AIC) values.

The AIC is given by $AIC = 2k - 2ln(\hat{L})$, where k is the number of estimated parameters and $ln(\hat{L})$ is the log likelihood function of the specified distribution.

The p-values are the size of the test or the probability of rejecting the null hypothesis when it is true. A small p-value leads to the rejection of the null hypothesis. The p-value is compared with a predetermined level of significance (LOS). If the computed p-value is less than the LOS, the null hypothesis is rejected at the specified LOS. The null hypothesis is the hypothesis to be tested. It is normally denoted as H_0 or H. In this paper, the LOS considered are 5% and 1%.

The AIC value is used to identify the model of best fit to the data. Among the models, the one which has the least value is considered as the best fit.

5. Results

Table 5 shows the test statistic and p-values for the three tests corresponding to the arrival pattern to the three banks. From the table, it is evident that the p-values of the test statistics SW and AD corresponding to Bank-I are less than 0.05 and 0.01. Thus, we reject the null hypothesis that the arrivals to the Bank-I follow normal distribution at 5% and 1% levels of significance.

Bank	SW Test		CVM Test		AD Test	
	W	p-value	W^2	p-value	A^2	p-value
Bank-I	0.84437	0.004296	0.4302	0.2066	1.242	0.002
Bank-II	0.93615	0.2026	0.10193	0.9727	0.509	0.175
Bank-III	0.92675	0.1337	0.18845	0.7553	0.585	0.110

Table 5: *Test statistic and p-value for testing normal distribution*

Tables 6-8 provide the summary of fitting the distribution to the number of customers who arrived at the three banks. The test used was the AD test. From the results, table 6 shows that the smallest extreme value distribution has a higher p-value compared to the rest. It is less than 0.05 but greater than 0.01. Hence it can be concluded that the data give enough evidence at 1% LOS to support that the smallest extreme value distribution is a good fit to the arrival pattern of customers to Bank-I. This is also evident from tables 7 and 8, corresponding to Bank-II and Bank-III and at a higher (i.e., 5%) LOS too. The p-values are greater than 0.25. Based on the same criteria, the next best distribution could be the Weibull distribution.

The software package also reports the AIC values and as depicted in tables 6-8, it is again clear that the smallest extreme value distribution has the least AIC value among all distributions. A slight difference of 0.5 on the higher side of the smallest extreme value distribution than the Weibull is seen in the AIC value, but this could be because of the number reporting to the banks. Despite this difference, it could still be concluded that the smallest extreme value distribution is a good fit for the arrival data followed by the Weibull.

Distribution	Log-Likelihood	AD	p - value	AIC
Smallest Extreme Value	-126.5	0.941	0.018	256.9
Weibull	-127.7	1.139	<0.01	259.4
Normal	-129.7	1.242	0.002	263.4
Lognormal- Three Parameter	-129.7	1.242	0.002	265.4
Gamma	-131.1	1.435	<0.005	266.3
Loglogistic	-131.4	1.280	<0.005	266.9
Lognormal	-132.0	1.541	<0.0001	268.0
Largest Extreme Value	-133.5	1.597	<0.01	271.0
Exponential- Two Parameter	-138.1	3.915	<0.001	280.3
Exponential	-155.8	6.361	<0.001	313.6

Table 6: *Distribution fitting summary for number of customers arrived at Bank-I*

Distribution	Log-Likelihood	AD	p - value	AIC
Smallest Extreme Value	-122.1	0.275	>0.25	248.2
Weibull	-122.3	0.335	>0.25	248.6
Normal	-123.5	0.509	0.175	250.9
Gamma	-124.1	0.621	0.109	252.3
Lognormal- Three Parameter	-123.5	0.509	0.175	252.9
Lognormal	-124.5	0.685	0.062	253.1
Loglogistic	-124.6	0.590	0.082	253.2
Largest Extreme Value	-126.4	0.893	0.023	256.7
Exponential- Two Parameter	-131.9	3.686	<0.001	267.8
Exponential	-155.5	6.955	<0.001	312.9

Table 7: *Distribution fitting summary for number of customers arrived at Bank-II*

Distribution	Log-Likelihood	AD	p - value	AIC
Smallest Extreme Value	-120.5	0.334	>0.25	245.0
Weibull	-120.6	0.399	>0.25	245.2
Normal	-121.8	0.585	0.110	247.6
Gamma	-122.4	0.691	0.075	248.7
Lognormal	-122.7	0.751	0.042	249.5
Lognormal- Three Parameter	-121.8	0.585	0.110	249.6
Loglogistic	-122.8	0.656	0.049	249.7
Largest Extreme Value	-124.6	0.978	0.015	253.2
Exponential- Two Parameter	-128.8	3.317	<0.001	261.5
Exponential	-155.2	7.104	<0.001	312.3

Table 8: *Distribution fitting summary for number of customers arrived at Bank-III*

6. Conclusion

On analyzing the data, it is evident that the minimum number of servers required for bank transactions is thirty-three at Bank-I, thirty-one at Bank-II and twenty-three at Bank-III. This seems to be a large number. Owing to the advancement of technology, the problem can be resolved as it is an optimization problem.

The focus of this paper was on the identification of a suitable distribution to the arrival pattern, and it is observed from the tests for normality as well as goodness of fit test that the smallest extreme value distribution is a good fit to the data.

The smallest extreme value distribution is also called the Gumbel distribution or type I extreme value distribution. It has two parameters, with location parameter as the mode. It belongs to the exponential family. It is a negatively skewed distribution and is platykurtic. Although the distribution is continuous, its discrete analog may be considered as the arrival process is a discrete setup. Further research in queueing theory can be done using the discrete analog of smallest extreme value distribution as an alternative to Poisson.

Data Availability

The dataset used is a secondary source and is publicly available at
https://doi.org/10.1016/j.dib.2018.05.101

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Management ownership and earnings management: application of matching techniques

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Abstract. This paper investigates the effect of managerial ownership on earnings management for the sample of European listed companies. To estimate the treatment effect of a manager who is also a controlling shareholder on earnings management, entropy balancing (EB), propensity score matching (PSM) and Mahalanobis distance matching (MDM) techniques are employed to identify a control sample. The EB method of reweighting control sample observations is compared to PSM and MDM techniques, as well as to standard ordinary least squares (OLS) regression, while controlling for the effects of company size, profitability, solvency, sales growth, and ownership concentration. The results demonstrate a significant positive relationship between management ownership and earnings management only when using the entropy balancing approach, and an insignificant relationship when applying PSM, MDM and OLS methods.

Keywords: earnings management, entropy balancing, Mahalanobis distance matching, management ownership, propensity score matching

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1. Introduction

The separation of ownership and control is usually argued to be the cause of information asymmetries and agency problems between managers (agents) and shareholders (principals). A pronounced information asymmetry problem usually leads to significantly higher transaction costs because managers can use their insider position to manipulate earnings [17]. Managerial participation in a company's ownership can mitigate agency problems by aligning the interests of owners and managers [26, 19]. However, high levels of managerial ownership may lead to managerial entrenchment, i.e. the additional voting power that permits them to secure their position in the company and protect them from specific disciplinary actions (Khan and Mather, 2013). Therefore, managerial ownership can also create agency problems through managerial entrenchment.

Prior studies have empirically confirmed a non-monotonic relationship between earnings management and managerial ownership [17]. Management ownership at low levels decreases the incentive to manipulate earnings, while management ownership at high levels increases earnings management [19, 16]. Namely, management ownership at low levels can reduce Type 1 agency conflict emerging from the separation of ownership from control. However, at high ownership

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levels, concentrated ownership can lead to Type 2 agency conflicts between controlling managers and non-controlling owners outside the company [3, 17, 2].

The main objective of this research is to analyse the level of earnings management in a specific company setting in which managers are also controlling shareholders, compared to similar companies that do not have managers as controlling shareholders. In order to identify the control sample, entropy balancing, propensity score and Mahalanobis distance matching techniques are adopted, and the estimated results are compared with the traditional OLS regression approach. Namely, traditional linear regression of an outcome variable on a binary treatment indicator with included control variables addresses endogeneity concerns only under the assumptions that there are no other unobserved factors that could confound inferences by affecting the treatment indicator; and that the relationships between the control variables and the independent variables are linear [37]. Matching techniques help address concerns regarding the assumption of a linear functional form in OLS regressions [37, 4, 33]. In addition, the matched-sample research design provides another advantage over traditional OLS regression design by restraining the extrapolation of estimated results beyond the distributional support from the data of the treatment and control samples, because linear regression assumes that model factors remain constant beyond the support of the data. This OLS regression assumption can result in producing biased inferences and measurement errors if applied to the sample with extreme values [25, 7]. As previous literature [19, 29] provides strong empirical evidence of non-linear relationships in models of management ownership and earnings management, an attempt is made to exploit the benefits of applying matching techniques to gain a new insight and to address methodological concerns.

The relationship between management ownership and earnings management is examined using a large sample of 2,525 European listed companies in 2019 and 2020. In order to estimate the treatment effect of a manager who is also a controlling shareholder on earnings management, standard OLS regression is first applied, with controlling variables for the effects of company size, profitability, solvency, sales growth, and ownership concentration. Afterwards, the estimated regression results are compared with the results of the most common matching techniques used in the literature [37]: propensity score matching (PSM), Mahalanobis distance matching (MDM), and entropy balancing (EB). Results indicate a statistically significant positive relationship between management controlling ownership and earnings management only when the entropy balancing approach is adopted.

This study makes several important contributions to the existing literature. First, previous research has documented a non-linear, U-shaped pattern relationship between the level of management ownership and earnings management. Prior empirical work is extended by analysing management ownership only at high levels, i.e. only in companies with managerial controlling ownership, and by adopting matching techniques. This approach accommodates non-linearity concerns and exploits other advantages of matched-sample research design over traditional linear regression design. Second, when analysing the management ownership impact on earnings management, ownership concentration is also accounted for. This allows for consideration of both Type 1 agency conflict emerging from the separation of ownership from control and for Type 2 agency conflicts arising from ownership concentration, i.e. conflicts between controlling and non-controlling owners. Third, this paper provides useful implementation guidance for the most popular matching techniques and analyses the key advantages and drawbacks of each matching tool.

The rest of the paper is organised as follows. Section 2 provides the theoretical background and literature review on managerial ownership and earnings management. Section 3 briefly describes matched-sample research methodology. Section 4 presents the research design and sample selection. In Section 5, the main empirical results are presented, and Section 6 concludes the paper.

2. Theoretical background and literature review

Since reported earnings are important information for decision making, managers may abuse their discretion in financial reporting and impact the wealth distribution in a way that maximises their own expected benefits at the expense of other stakeholders. The flexibility in the accounting rules and the use of subjective assessments allow managers opportunistic behaviour in the preparation of financial statements. Financial reporting rules and standards established by standard-setters (the International Accounting Standards Board – IASB and the Financial Accounting Standards Board – FASB) allow managers to choose between different financial reporting policies and to make subjective assessments that can have direct influence on reported earnings. This set of financial reporting rules is defined ex ante and is generally accepted by all contracting parties. However, within a prescribed set of financial reporting rules, a certain degree of freedom of choice must exist, as it is not possible to propose rules for every possible situation [12]. Additionally, the choice of financial reporting policy can be an valuable decision-making information for users of financial statements.

Managers can use their discretion in choosing financial reporting policies either to maximise the wealth of all counterparties or the wealth of some counterparties. If management's choice of accounting policy is primarily aimed at increasing its own ex-post wealth by redistributing the wealth of other contracting parties, such behaviour is called opportunistic [39]. The divergence between management and owner interests is mainly caused by the separation of control and ownership, i.e. the Type I agency problem [26, 30]. One solution to reduce this conflict is to incorporate managers in the ownership structure of the company and align their incentives with those of the other shareholders. Contrary to the alignment hypothesis, the entrenchment hypothesis, first proposed by Morck et al. [27], states that an increase in ownership could enable managers to secure their position in the company and protect them from specific disciplinary mechanisms that could consequently lead to greater opportunity for opportunistic behaviour and earnings management [19, 16].

These two opposing hypotheses have been frequently empirically tested in the context of the relationship between financial reporting quality and management ownership. Although the majority of previous studies have found that an increase in managerial share ownership decreases the level of earnings management [38, 14], several papers have documented an insignificant [22, 13, 23] or a positive relationship between earnings management and managerial ownership [16]. Khan and Mather [19] and O'Callaghan et al. [29] provide evidence that the relationship between discretionary accruals (earnings management) and management ownership has a non-linear, U-shaped pattern. Khan and Mather [19] conclude that a negative relation between managerial ownership and discretionary accruals is found at lower levels of ownership, supporting the incentive alignment hypothesis and a positive relationship can be seen at higher level of ownership indicating managerial entrenchment. Furthermore, when a manager concurrently holds the position of majority shareholder, there is a significant incentive for tax evasion through earnings management. This dual role creates a unique dynamic in which the interests of management and ownership are closely aligned, potentially fostering more aggressive tax avoidance strategies. Empirical evidence suggests that concentrated ownership, particularly involving manager-shareholders, is correlated with elevated levels of tax avoidance [18].

In addition to agency conflicts between owners and managers, another type of agency conflict (Type II agency problem) can occur between controlling and non-controlling shareholders [11, 30]. This type of conflict is more prevalent in less developed countries, where the ownership of listed companies is concentrated in a single shareholder. When a manager is also a majority shareholder, the agency problem shifts from a manager-owner conflict to a conflict between majority and minority shareholders. Some studies have examined the extent of earnings management in family firms, i.e. firms controlled and managed by family members. Paiva et al. [30] concluded that previous studies based on samples from the US and Western Europe have found evidence of higher quality financial reporting in family firms (e.g. [2, 5, 1]. In contrast, studies based on samples of companies from countries outside the US and Western Europe have demonstrated that family businesses are associated with lower quality of financial reporting (higher levels of earnings management) compared to non-family businesses [10, 6].

To conclude, there is a lack of studies on the managerial ownership and earnings management that can address both Type 1 agency conflict, arising from the separation of ownership from control, and Type 2 agency conflicts, arising from the ownership concentration. This paper aims to fill the gap and, in accordance to previous literature, assumes that managerial controlling ownership will be positively related to earnings management.

3. Matched-sample research methodology

The general idea behind matching design is to estimate the treatment effect by identifying a treatment sample and a control sample of observations that are as similar as possible in terms of underlying covariates. Two main advantages of matched-sample design over standard linear regression are the avoidance of an assumption of a linear functional form in linear regression and the restriction of the extrapolation of estimations beyond the distributional support of the data from the combined control and treatment samples [37, 25, 7]. When there are some balance and overlap problems between treatment sample and control sample in the dataset, this suggests regression adjustment would rely on extrapolation.

The main goals are to ensure almost exact covariate balance between the control and treatment sample and to restrict the data only to a region of overlap between the samples. However, the matching methods do not address other endogeneity concerns regarding the control for unobserved confounding factors, and they also have some drawbacks. For instance, it is very difficult to find an exact control match for each treatment sample observation, especially if there are many continuous covariates. The most commonly used matching techniques – propensity score matching, Mahalanobis distance matching and entropy balancing, are briefly described in this section.

The propensity score matching is based on the main assumption of strong ignorability, which states that, conditional on observable covariates, the assignment to the treatment or control groups is independent of potential outcomes (the unconfoundedness condition) and that every observation has a positive probability of being assigned to the treated or control group conditional on observable covariates (the overlap condition) [31, 28]. The propensity score represents the likelihood of an observation being assigned to a treatment group conditional on observed variables and it is commonly used to identify observations that have similar characteristics but differ only in treatment assignment [28]. The propensity score matching approach is usually applied in four steps [28]. In the first step, logit or probit models are used to estimate treatment probabilities based on observed variables (see [36], for practical guidance on logistic regression modelling). In this study, a logit model is utilized to estimate the treatment probability, as it is acknowledged as the most prevalent method for estimating propensity scores in the accounting literature [15, 24]. Additionally, visual inspections and tests for the normal distribution of residuals are conducted, and the goodness-of-fit for both logit and probit models is compared to determine the optimal model. In the second step, the estimated propensity scores from the first step are used to find pairs of observations in the treatment and control groups with similar propensity scores. Several alternative algorithms and design choices can be employed for this purpose: matching with or without replacement, the number of control observations for each control observation, the choice of caliper width, the inclusion of nonlinear terms in the model, and the choice of matching algorithm. These subjective design choices are considered the greatest limitations of the PSM methodology because these choices can influence on the matched sample composition and, consequently, affect the derived conclusions [20, 9]. One of the most popular algorithms for matching is nearest neighbour matching, that finds one or more observations in the control group with the closest propensity score for each observation in treatment group (1:1 or 1:n). An alternative algorithm (optimal matching) considers the smallest average absolute differences for the whole sample [28]. Besides, Narita et al. [28] state that matching can be with or without replacement, depending on whether a controlled matched observation is reintroduced into the control group for next-round of matching or not. Matching with replacement can be useful when the pool of control observations is smaller or not considerably larger than the treatment sample [37]. However, matching with replacement can cause that only a several control observations are used many times for matching (i.e. receive extremely large weights), and lead to inferences that are much more dependent on a single observation [37, 25]. Finally, matching can also be limited to not exceed a maximum propensity score distance (i.e. caliper width) for matched pair. In this research design, the nearest neighbour matching algorithm (1:1) is applied with a maximum caliper distance in propensity scores between matched pairs of 0.01, and control observation is allowed to be matched with only one treatment observation (matching without replacement). The predominant method of matching in accounting research is "one-to-one" matching, where each treatment observation is paired with a single control observation [33]. However, one-to-many matching may be beneficial when there is limited common support and/or the pool of control observations is not significantly larger than the treatment sample [37]. In this study, a large control sample and high common support are present.Therefore, one-to-one matching with a strict caliper distance of 0.01 is employed. After finding a pair control observation for each treatment observation, the next step is to check whether the covariate distributions are balanced between the treatment and the control group. The common approach is to use a two-sample t-test to assess significant differences in covariate means between the control and treatment group or to use the standardised bias, that estimates the distance in marginal distribution of covariates [28]. The fourth step in the analysis is to estimate the impact of the treatment variable on the dependent variable, usually by conducting a multiple regression analysis. Regression analysis controls for additional factors that could affect the outcome variable after treatment, and can also serve as a double-robust estimation that combines regression adjustment and propensity score weighting to provide a more reliable estimate.

Propensity scores reduce the multidimensional covariate space into a single scalar value. Consequently, two units with similar propensity scores may still have substantially different covariate profiles. This emphasizes a key consideration that achieving balance on the propensity score does not ensure balance across all individual covariates [21]. Mahalanobis distance matching (MDM) is a method similar to PSM, which tends to produce closer pairs on all covariates, whereas PSM may yield balanced samples overall but not necessarily for individual pairs. Propensity score matching (PSM) is typically favoured for larger samples with numerous covariates, whereas Mahalanobis distance matching (MDM) tends to be more effective for smaller samples with fewer covariates [35].

A popular alternative approach to score and Mahalanobis distance matching is entropy balancing. The most important advantages of entropy balancing over PSM is that entropy balancing has fewer subjective design choices and can eliminate all differences in covariates between treatment and control observations, whereas in PSM design, random differences still remain after matching [37]. In entropy balancing, control observations receive weights between zero and one, and the procedure optimizes weights to achieve an exact balance between the treatment and control sample in terms of covariates. The researcher can choose to balance distributions based on the first moment (i.e. on means), the second moment (i.e. on variances), or even the third moment (i.e. skewness). In the present study, entropy balancing is performed to reweight control sample observations to equalize the first (mean) and second (variance) moments of the distributions. The entropy balancing follows the same four steps for application analogous to propensity score matching. However, entropy balancing has fewer subjective design choices and manages to achieve a much more precise balance in covariates between the treatment and control sample than propensity score matching. Despite that, similar to PSM with replacement, it can cause that a few control observations can receive extremely large weights, making inferences more sensitive to specific observations.

4. Research design and sample selection

The relationship between management ownership and earnings management is analysed on a large sample of 2,525 European listed firms in the period 2019–2020. Since the calculation of most variables requires data from the previous year, the initial sample is restricted to active, publicly listed companies that have available accounts for 2018, 2019, and 2020 and that apply IFRS standards. The final sample consists of 2,525 companies. All necessary data is collected from the BvD Orbis Europe database.

First, a standard OLS regression is conducted with controlling variables for the effects of company size, profitability, solvency, sales growth, and ownership concentration. Afterwards, the estimated regression results are compared with the results of propensity score matching and entropy balancing approach.

The dependent variable represents the level of earnings management and it is measured by estimating cross-sectional discretionary accruals from the modified Jones model [8], which is the most commonly used proxy for earnings management. A higher proportion of discretionary accruals in earnings implies more earnings management or lower earnings quality.

The modified Jones model [8] for computing cross-sectional discretionary accruals is estimated in two stages:

$$\frac{\mathrm{ACC}_{ijt}}{\mathrm{TA}_{ijt-1}} = \beta_{0jt}\left(\frac{1}{\mathrm{TA}_{ijt-1}}\right) + \beta_{1jt}\left(\frac{\Delta\mathrm{REV}_{ijt}}{\mathrm{TA}_{ijt-1}}\right) + \beta_{2jt}\left(\frac{\mathrm{GPPE}_{ijt}}{\mathrm{TA}_{ijt-1}}\right) + \varepsilon_{it}, \tag{1}$$

$$\mathrm{DACC}_{ijt} = \frac{\mathrm{ACC}_{ijt}}{\mathrm{TA}_{ijt-1}} - \left[\hat{\beta}_{0jt}\left(\frac{1}{\mathrm{TA}_{ijt-1}}\right) + \hat{\beta}_{1jt}\left(\frac{\Delta\mathrm{REV}_{ijt}}{\mathrm{TA}_{ijt-1}}\right) + \hat{\beta}_{2jt}\left(\frac{\mathrm{GPPE}_{ijt}}{\mathrm{TA}_{ijt-1}}\right)\right]. \tag{2}$$

where the subscript i represents each company in the industry-year estimation portfolios j by two-digit SIC codes, ACC is total accruals, TA is total assets at the beginning of the year, REV is the change in revenue, GPPE is property, plant and equipment, and DACC is discretionary accrual component. A minimum of 10 observations is required for each industry-year portfolio.

The absolute value of DACC (*Abs dac*) is used as earnings management proxy, where higher values of *Abs dac* represent a higher likelihood of earnings management.

The main treatment variable is management controlling ownership (*CSH M*), which is equal to one if a company's manager is a controlling shareholder, and zero otherwise. Besides, additional controlling variables are included for the effects of company size (*Size*), profitability (*ROA*), solvency (*SR*), sales growth (*Saleg*), and ownership concentration (*BVDind*). To account for the primary motives behind manipulations, and in accordance with the majority of previous studies (e.g., [24]), control variables for company size (*Size*), profitability (*ROA*), solvency (*SR*), and sales growth (*Saleg*) are included. These motives are derived from Positive Accounting Theory, which comprises three hypotheses: the bonus plan hypothesis, the debt covenant hypothesis, and the political cost hypothesis [39]. These hypotheses typically serve as the foundation for earnings management practices. Variables descriptions are provided in Table 1.

Variable	Description
CSH_M	1 if a company manager is a controlling shareholder, and zero otherwise
dac	Cross-sectional discretionary accruals from the modified-Jones model [8] for each year using all company-year observations in the same two-digit SIC code
Abs_dac	The absolute value of the dac
SR	Solvency ratio, shareholders' funds divided by total assets multiplied by 100
Size	The natural logarithm of total revenues
ROA	Return on assets, profit or loss before tax scaled by average total assets multiplied by 100
Saleg	Sales growth, total sales in year t divided by total sales in year $t-1$
BVD_ind	The BvD ownership concentration indicator has five levels: – **A**: Low (no shareholder >25%) – **B**: Medium-low (25.1%-50%) – **C**: Medium-high (≥50.1%) – **D**: High (>50% with branches) – **U**: Other cases

Table 1: *Variables definitions*

The following model is used to estimate both ordinary least squares regression and weighted ordinary least square regression, where weights are used from propensity score matching, Mahalanobis distance matching, and entropy balancing:

$$\text{Abs dac}_{it} = \beta_0 + \beta_1 \text{CSH M}_{it} + \beta_2 \text{SR}_{it} + \beta_3 \text{Size}_{it} + \beta_4 \text{ROA}_{it} + \beta_5 \text{Saleg}_{it} + \sum \beta \text{BVD ind}_{it} + e_{it} \tag{3}$$

5. Results

In the first part of the empirical analysis, summary descriptive statistics are presented for the full sample and for subsamples of companies with and without managers as controlling shareholders (Table 2). The primary objective of this part of analysis is to assess whether the control sample (CSH M=0) serves as a valid counterfactual for the treatment sample (CSH M=1) or if there is an imbalance in covariates across these samples.

Results from Table 2 indicate that companies with controlling managerial ownership tend to have higher level of earnings management (Abs dac), higher profitability (ROA), but lower size (Size), level of solvency (SR), and revenue growth (Saleg). The statistical significance of these differences is tested in Table 5.

Table 3 presents the distribution of companies by BvD ownership concentration for the full sample. The majority of companies in the sample have high ownership concentration, with a shareholder who owns more than 50% voting rights.

Correlation analysis is presented in Table 4. Spearman's rank correlations are shown above the diagonal and Pearson's correlation coefficients are below the diagonal. The estimated results indicate a significant correlation between earnings management (Abs dac) and all other variables, except for the managerial controlling ownership indicator, which is only marginally significant (at 10% level).

Variable	N	Mean	SD	p25	Median	p75
Panel A: Control subsample (CSH_M = 0)						
Abs_dac	3619	0.055	0.058	0.016	0.037	0.071
Size	3594	18.736	2.704	16.99	18.805	20.589
ROA	3591	0.641	15.549	-1.593	3.299	7.401
SR	3578	43.331	25.131	29.635	43.431	59.117
Saleg	3597	1.213	3.976	0.892	1.011	1.112
Panel B: Treatment subsample (CSH_M = 1)						
Abs_dac	417	0.059	0.062	0.014	0.040	0.077
Size	414	18.239	2.019	16.986	18.185	19.452
ROA	416	1.642	12.532	-0.982	3.439	7.137
SR	416	40.779	25.677	28.121	40.709	56.292
Saleg	415	1.026	0.389	0.864	0.996	1.103
Panel C: Full sample						
Abs_dac	4036	0.055	0.058	0.016	0.037	0.072
Size	4008	18.684	2.646	16.990	18.737	20.437
ROA	4007	0.744	15.265	-1.556	3.309	7.386
SR	3994	43.065	25.198	29.502	43.117	58.804
Saleg	4012	1.194	3.768	0.890	1.010	1.112

Table 2: *Descriptive statistics*
Note: See Table 1 for variable definitions.

BVD_ind	Frequency	Percent	Cumulative
A	1,215	30.10	30.10
B	1,221	30.25	60.36
C	117	2.90	63.26
D	1,407	34.86	98.12
U	76	1.88	100.00
Total	4,036	100.00	

Table 3: *Distribution by BvD ownership concentration indicator*
Note: See Table 1 for variable definitions.

	CSH_M	Size	Abs_dac	ROA	SR	Saleg
CSH_M	1.000	-0.078	0.018	0.004	-0.036	-0.028
		(0.000)	(0.272)	(0.803)	(0.023)	(0.079)
Size	-0.065	1.000	-0.208	0.271	-0.273	0.035
	(0.000)		(0.000)	(0.000)	(0.000)	(0.029)
Abs_dac	0.026	-0.214	1.000	-0.087	-0.047	0.024
	(0.099)	(0.000)		(0.000)	(0.003)	(0.128)
ROA	0.014	0.336	-0.162	1.000	0.273	0.278
	(0.374)	(0.000)	(0.000)		(0.000)	(0.000)
SR	-0.036	-0.225	-0.097	0.251	1.000	0.034
	(0.022)	(0.000)	(0.000)	(0.000)		(0.032)
Saleg	-0.015	0.002	0.032	0.001	-0.001	1.000
	(0.339)	(0.905)	(0.046)	(0.962)	(0.970)	

Table 4: *Pearson/Spearman correlation matrix*
Note: See Table 1 for variable definitions. Pearson's correlation coefficients are below the diagonal and Spearman's rank correlations are above the diagonal. P-values are in parentheses.

Table 5 provides evidence of a significant covariate imbalance between the control and treatment sample, especially in the value of Size and the Solvency ratio.

Variable	N Ctrl	N Trt	Mean Ctrl	Mean Trt	Diff	SE	t	p
Abs_dac	3,619	417	0.054	0.059	-0.005	0.003	-1.50	0.140
ROA	3,591	416	0.640	1.642	-1.002	0.790	-1.25	0.206
SR	3,578	416	43.331	40.779	2.551	1.305	1.95	0.051
Saleg	3,597	415	1.213	1.026	0.188	0.196	0.95	0.338
Size	3,594	414	18.736	18.239	0.497	0.137	3.65	0.001

Table 5: *T-test differences in mean values between treatment and control subsamples before balancing*
Note: See Table 1 for variable definitions. Ctrl = Control, Trt = Treatment, SE = Standard Error

To address covariate imbalance concerns across samples with and without managerial controlling ownership, propensity score matching (PSM), Mahalanobis distance matching (MDM), and entropy balancing (EB) are employed to reweight observations and to achieve covariate balance. In order to apply PSM and obtain the propensity score, a logit model is first estimated, where the managerial controlling ownership indicator (CSH M) is the dependent variable, and five covariates are included in the model as independent variables (Size, ROA, BVD ind, SR, and Saleg). The estimated propensity scores (Table 6) indicate that Size, Profitability, BvD ownership concentration, and Solvency ratio are significant covariates in the propensity score model. The Hosmer-Lemeshow test was also performed to find evidence that the estimated logit model adequately fits the data.

CSH_M	Coeff.	St.Err.	t-value	p-value	Sig
Size	0.833	0.023	-6.58	0.000	***
ROA	1.013	0.005	2.36	0.018	**
BVD_ind	1.651	0.345	2.40	0.016	**
SR	0.992	0.002	-3.39	0.001	***
Saleg	1.042	0.098	0.43	0.665	
Constant	14.451	7.997	4.83	0.000	***
Number of obs: 1,502					
McFadden's pseudo r-squared: 0.030					
Chi-square: 52.788					
Prob > chi2: 0.000					
Hosmer–Lemeshow chi2(8): 11.54					

Table 6: *Propensity score model*
*Note: See Table 1 for variable definitions. Standard errors are in parentheses. *** p<0.01, ** p<0.05, * p<0.1*

The distribution of propensity scores from the logit model in Table 6 is presented graphically in Figure 1 to examine the appropriate distributional overlap in observable covariates from the treatment and control subsamples. Figure 1 indicates adequate distributional overlap and shows that the estimated propensity scores are not concentrated around the extreme values of one and zero.

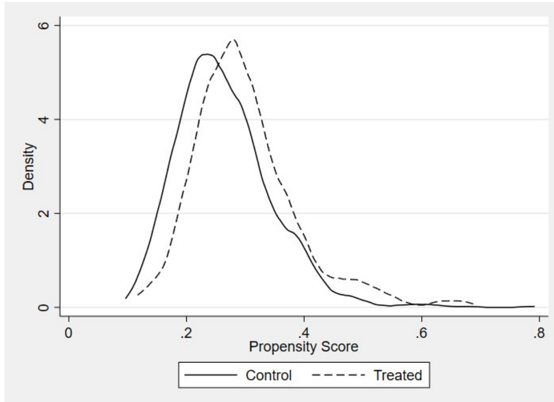
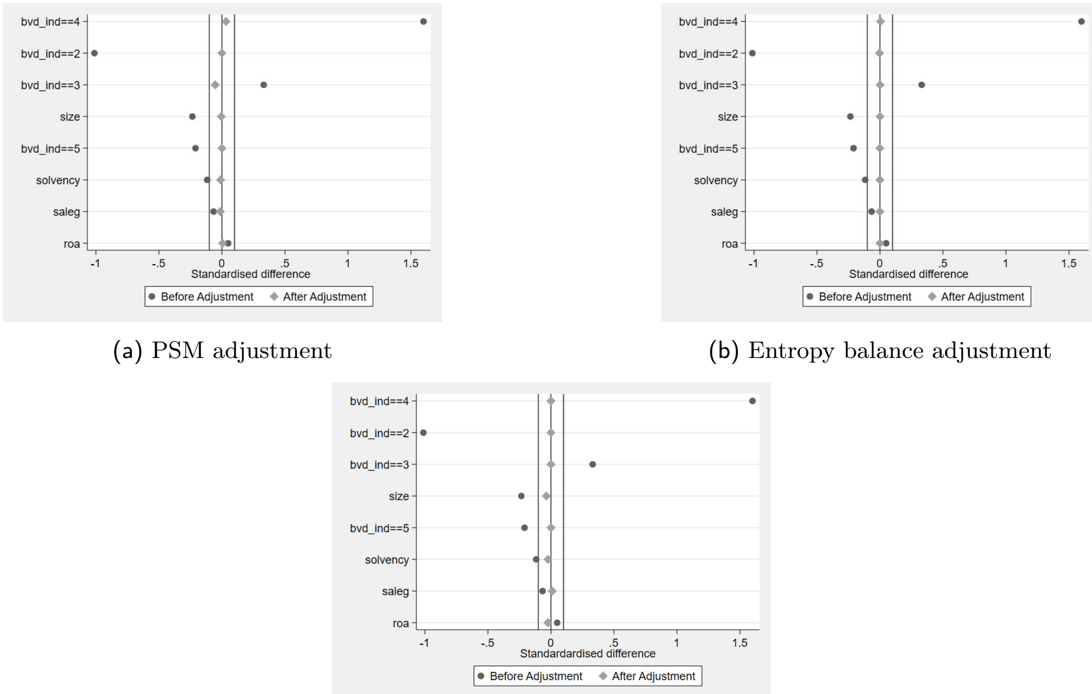


Figure 1: *Propensity score density for the treatment sample (CSH M=1) and the control sample (CSH M=0).*

Variable	Mean treated	Mean untreated	Std. diff.
Panel A: Before balancing			
Size	18.24	18.80	-0.234
ROA	1.77	1.09	0.049
SR	40.49	43.43	-0.117
Saleg	1.03	1.22	-0.066
Panel B: PSM			
Size	18.36	18.37	-0.005
ROA	1.84	1.76	0.005
SR	41.30	41.57	-0.011
Saleg	1.03	1.07	-0.013
Panel C: Mahalanobis			
Size	18.24	18.33	-0.037
ROA	1.77	2.09	-0.024
SR	40.49	41.07	-0.023
Saleg	1.03	1.00	0.010
Maximum weight			6.00
Panel D: Entropy balancing			
Size	18.24	18.24	0.000
ROA	1.77	1.77	0.000
SR	40.49	40.49	-0.000
Saleg	1.03	1.03	-0.000
Maximum weight			2.974

Table 7: *Balance statistics before/after matching*
Note: See Table 1 for definitions.

Observations are matched using the nearest neighbour algorithm (1:1) with a maximum caliper distance in propensity scores between matched pairs of 0.01, allowing each control observation to be matched with only one treatment observation (matching without replacement), as recommended by the majority of previous studies [33, 37, 25]. The propensity score matching approach resulted in a sample of 800 matched observations (400 from both the treatment and control subsamples).



(a) PSM adjustment

(b) Entropy balance adjustment

(c) Mahalanobis matching

Figure 2: *Standardised differences in means before and after matching.*

Table 7 shows that PSM, MDM, and EB approaches were successful in balancing covariates, since the standardised differences in means fall within acceptable boundaries for balanced covariates (i.e., inside of the values of +/-0.1, according to [32, 25]. However, the entropy balancing approach (by the definition) generates a more exact balance in covariates between the treatment and control sample than propensity score matching, and the maximum weight to control observation in EB is 2.974, suggesting that estimations are not (over)sensitive to a specific observation.

Standardised differences in means before and after matching for PSM, EB, and MDM are also presented graphically in Figure 2. Standardised difference is calculated as the difference in means between the treatment and control subsamples divided by the square root of the average variance of treatment and control subsample.

Finally, Table 8 presents the results of estimating the main research model from Eq. (3) by using ordinary least squares (OLS) in model (1), entropy balancing on the first moment weighted regression (2), entropy balancing on the second moment weighted regression (3), regression using propensity score matched observations (4), and regression using Mahalanobis matched observations (5).

The results provide evidence of a significant positive relationship between managerial controlling ownership and earnings management only when using the entropy balancing on the first moment approach, while this relationship is not significant when applying the PSM, MDM, and OLS methods.

Abs_dac	**(1) OLS**	**(2) EB (1st)**	**(3) EB (2nd)**	**(4) PSM**	**(5) Mahal.**
CSH_M	0.003	0.004**	0.002	0.004	0.000
	(0.003)	(0.002)	(0.002)	(0.004)	(0.004)
Size	-0.005***	-0.004***	-0.006***	-0.006***	-0.006***
	(0.000)	(0.000)	(0.000)	(0.001)	(0.001)
ROA	0.000***	0.000	0.000***	0.000	0.000**
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
BVD_ind(1)					
BVD_ind(2)	0.000	-0.010	0.001		
	(0.002)	(0.058)	(0.338)		
BVD_ind(3)	-0.005	-0.002	0.003		
	(0.005)	(0.042)	(0.161)		
BVD_ind(4)	-0.002	0.000	0.002	0.002	0.011
	(0.002)	(0.041)	(0.161)	(0.007)	(0.007)
BVD_ind(5)	0.003	0.008	0.006		
	(0.006)	(0.168)	(1.196)		
SR	0.000***	0.000***	0.000***	0.000***	0.000***
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Saleg	0.000**	0.005***	0.022***	0.005*	0.029***
	(0.000)	(0.001)	(0.002)	(0.003)	(0.007)
Constant	0.159***	0.133***	0.146	0.179***	0.140***
	(0.008)	(0.042)	(0.162)	(0.022)	(0.024)
Observations	3,948	3,948	3,948	800	708
R-squared	0.071	0.050	0.077	0.084	0.088

Table 8: *Regression results (OLS, EB weighted, PSM, and Mahalanobis matched)*
*Note: Standard errors in parentheses. *** $p<0.01$, ** $p<0.05$, * $p<0.1$.*

6. Conclusion

This research adopts entropy balancing, propensity score, and Mahalanobis distance matching techniques to investigate the effect of managerial controlling ownership on earnings management for the large sample of European listed companies. The estimated results suggest that managerial controlling ownership is significantly positively related to earnings management only when using entropy balancing on a first moment (mean) approach. However, our analysis did not find a significant relationship between these variables when applying alternative matching techniques, which limits the influence and generalizability of the gathered evidence. The temporal scope of the study, covering 2019–2020 also raises concerns regarding generalisation because the COVID-19 pandemic significantly disrupted economic activity in 2020 [34], potentially affecting earnings management behaviour due to heightened uncertainty and altering ownership structures. Additionally, while matching methods address observable confounders, the potential impact of unobservable factors always remains a concern in this type of archival research studies.

Even though there are many previous studies on managerial ownership and financial reporting quality, prior research has not accounted for both Type 1 agency conflict, emerging from the separation of ownership from control, and Type 2 agency conflicts, arising from ownership concentration. This paper tries to extend previous empirical work by analysing the management ownership only at high levels, i.e. only companies with managerial controlling ownership, and by adopting matching techniques. In this way, we can accommodate non-linearity concerns and exploit other advantages of a matched-sample research design over a traditional linear regression design. Additionally, by controlling for ownership concentration, we can account for both Type 1 agency conflict, arising from the separation of ownership from control, and for Type 2 agency conflicts, arising from the ownership concentration, i.e. conflicts between the controlling and non-controlling shareholders. Besides, this paper provides useful implementation guidance for the most popular matching methods: propensity score matching, Mahalanobis distance matching, and entropy balancing. We underscore that the design choices inherent in various matching techniques can significantly affect sample composition and the estimated findings. Consequently, we recommend that future researchers transparently disclose their design choices and complement matched sample results with alternative research designs. Moreover, our findings bear substantial practical implications for corporate governance in Europe. For instance, regulators and minority shareholders may interpret pronounced earnings management by controlling managers as an indicator to bolster oversight mechanisms, enhance transparency, or revise governance policies. Financial analysts and auditors could also leverage our findings to refine their audit procedures and analytical approaches when evaluating firms with controlling managers.

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Determinants of real estate prices in Croatian counties: spatial panel data analysis

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Abstract. This paper examines the determinants of real estate prices across Croatian counties, taking into account not only economic and demographic factors, but also tourism-related variables. The main focus is on spatial spillover effects, which are based on the assumption that real estate prices in one county are influenced by the developments in neighboring counties. The study covers the period from 2011 to 2019 and confirms that real estate prices are influenced by both local and regional factors. The key determinants include the number of tourist accommodation units, population size, and GDP per capita. An important result is the existence of a statistically significant spatial dependence across all specifications. This indicates that changes in real estate prices extend beyond county boundaries, underscoring the interconnectedness of the real estate market. These results underline the need for using spatial econometric models, which provide a more comprehensive understanding of regional price dynamics by capturing spillover effects between counties.

Keywords: counties, real estate prices, spatial panel data, spillovers, tourism

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1. Introduction

The relationship between tourism and real estate prices has become the focus of research interest due to the growing availability of travel, which has led to an increase in tourist numbers and a greater impact of tourism on the economy in many countries. In addition, the advances in empirical methods of analysis, particularly the application of spatial econometrics, have further contributed to the study of these issues [17, 9, 3, 8]. The recent surge in real estate prices in Europe requires a more detailed analysis of the factors influencing the development of real estate prices. This study focuses on the Republic of Croatia, specifically at the county level, where real estate supply and demand largely depend on economic, demographic, and tourism-related factors [13, 12, 2, 8, 17, 9, 7, 18]. Typical economic variables examined in the literature are GDP per capita, average net income (wages), interest rate, unemployment rate, credit supply, and the purchasing power of the population. Demographic indicators typically include the number of new businesses opened, the number of new residents moving into the area and the number of building permits issued. In recent research, tourism variables are increasingly included as they

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have a significant impact on the economy [14, 16, 13, 17, 9]. Among the tourism indicators, the share of rental properties, the share of private accommodation in total tourist capacity, the length of the summer tourist season, the number of overnight stays by tourists, and the length of their stay in the country show the most important effects on real estate prices [17, 9, 3].

This study analyzes all counties of the Republic of Croatia, taking into account their mutual interconnections. The aim of the study is to examine the impact of tourism-related indicators on the real estate prices in each county and to employ spatial panel models to demonstrate how tourism significantly contributes to the transmission of real estate prices across county borders. Due to the county-level focus, variables available only at the national level, such as the purchasing power of the population, interest rates on housing loans and the supply of housing loans, are excluded from the research, as their effects cannot be specifically isolated at the county level. Tourism indicators are widely recognized in the literature as key drivers of real estate price transmission between more and less tourism-developed regions. While the theoretical foundations of spatial price transmission have been established [15], empirical research on this phenomenon in Croatia remains limited. The first empirical analysis confirming this hypothesis for Croatia was conducted in 2023. [17] applied spatial econometrics to show that real estate prices in tourist cities and municipalities in the Republic of Croatia influence prices in less tourist-developed areas. However, with the exception of this paper, most previous studies have relied on traditional econometric methods, which cannot fully capture the spatial interdependencies between regions. These models typically overlook the potential spillover effects, where tourism-induced price increases in one region may affect neighboring areas. However, real estate markets are inherently spatial, as real estate prices depend not only on local supply and demand, but also on factors in neighboring regions. This is particularly true in the context of Croatian counties, which are linked by commuter flows, investments, shared infrastructure, and tourism-driven real estate demand. Ignoring these spatial interdependencies can lead to misleading conclusions about the determinants of real estate prices. Incorporating a spatial dimension into the analysis enables the explicit modeling of spillover effects, i.e. how shocks in one location may spread to other locations, improving the accuracy of the estimates and increasing their policy relevance. By applying spatial panel models, this study, therefore, provides a more sophisticated and robust analysis that allows us to more accurately capture the spatial transmission of real estate prices and provide deeper insights into the way tourism affects the Croatian real estate market.

The paper contributes to the literature by analyzing the impact of tourism on real estate prices in the Republic of Croatia at the county level, which has not been systematically studied before. In contrast to previous studies that have focused on the impact of tourism at the national level or within individual cities, this research examines counties as the primary spatial units of analysis. This approach provides valuable insights into regional macro-trends that remain invisible at narrower geographical levels. In addition, spatial econometric methods are used to investigate potential spatial spillover effects between counties, which have hardly been studied in the Croatian context. To this end, three different spatial econometric models and three different spatial matrices are applied, which underline the importance of spatial dynamics for the understanding of real estate price movements.

2. Determinants of real estate prices in Croatia: a literature review

Since the early 2000s, the development of real estate prices in Croatia has attracted increasing attention. Initially, price changes were well explained by traditional supply and demand indicators identified in the empirical literature. However, as time progressed, these traditional factors began to diverge from actual price movements, prompting researchers to investigate new determinants. Due to their strong and increasing correlation with real estate market dynamics, tourism-related variables gradually became recognized as key drivers of real estate prices —

not only in Croatia, but also in other (Mediterranean) countries. In this context, the literature highlights several important supply-side determinants of real estate prices, including the housing stock, the number of newly established businesses, the number of tourist accommodations, the share of private accommodations in the total supply, and the length of the tourist season. In addition, the number of overnight stays per capita, the number of tourist arrivals and the length of stay are also considered important indicators of the impact of tourism on real estate supply and pricing [16, 17, 9, 13, 2]. These variables reflect the capacity of the market and its ability to respond to increased tourism demand, which can have a significant impact on the availability and price of housing units.

On the other hand, the demand for real estate is determined by the income of the population, the supply of housing loans, the proportion of new immigrants, the purchasing power of the population, unemployment rate, interest rates, and GDP per capita [14, 13, 16, 17, 9, 10] These economic indicators reflect the population's financial ability to purchase or rent existing properties, as well as their ability to access financing sources such as loans.

The influence of these supply and demand factors has been examined by numerous authors, whose empirical results provide a deeper insight into the Croatian real estate market. The following section presents selected studies that have made a significant contribution to our understanding of the relationship between economic, demographic, and tourism-related variables and real estate prices in Croatia.

[16] analyzed the effects of various variables on real estate prices in Croatia from 1965 to 2003. The variables included were housing stock, income, availability of credit (as measured by interest rates), cost of living, and the share of the informal economy. He concluded that real estate investment was often driven by the preservation of wealth in an unstable economy rather than by the actual housing needs. This phenomenon, combined with inefficient housing use and insufficient investment in entrepreneurship, contributed to rising rental prices and declining average household income over the long term. [13] analyzed the period from 2005 to 2018 for 11 Central and Eastern European (CEE) EU Member States. The focusing on the impact of tourism and the number of newly established companies on real estate prices. Their findings indicate that an increase in the number of short-term rental apartments and a decrease in newly established companies correlate with higher property prices. It was also found that, tourism—especially tourism driven by foreign visitors—was a significant driver of demand in the real estate market.

At the local/national level, [9] analyzed the impact of tourism on housing affordability in 242 Croatian cities and municipalities between 2012 and2018. Their variables included seasonality, income, loan repayment capacity, number of tourist accommodations and tourism concentration. Their findings showed that extended summer season and a higher proportion of rental housing significantly reduce housing affordability and encourage population migration from tourist to non-tourist areas.

[10] theoretically considers credit supply and collateral as the most important variables. Croatia's accession to the EU has increased foreign demand for real estate. In 2017–2018, the prices rose by approximately 5 %, but without an increase in the volume of credit, which indicates that tourism was the main driver of price growth.

Finally, [14] used Croatian data from 2002 to 2022 to investigate how household income, interest rates, credit supply, GDP and the number of building permits influence house price risks using quantile regression. Their results underline the importance of macroeconomic and financial factors in shaping house price dynamics and affordability.

While most existing studies focus on local economic, demographic and tourism-related factors, spatial effects remain under-researched. When analyzing real estate prices at the county level, it is crucial to consider spatial interdependence, as price dynamics in one county can be influenced by developments in other counties. [7], one of the seminal studies in this area, examined regional house prices in the UK and identified the so-called "ripple effect", i.e., a phenomenon whereby price changes originate in one region, e.g. the South-East, and spread outwards. This finding suggests that regional real estate markets are interconnected and that price movements can spread spatially over time. According to [7], the most important mechanisms for this effect are migration, capital transfer, spatial arbitrage, and exogenous shocks.

In the real estate literature, there are other price transmission mechanisms between locations. According to [7, 4] the spatial spillover effect is a more comprehensive concept than the ripple effect. Unlike the ripple effect, spatial spillover effects refer more generally to the broader influence of economic or demographic changes in one region on surrounding regions. Spatial spillover effects can emerge through various mechanisms, e.g. spatial autocorrelation, contagion effects, and diffusion processes. In the Croatian context, these spillover effects can be particularly pronounced between counties due to their geographical proximity, migration between counties, similar economic characteristics, and factors such as shared infrastructure.

Despite the recognized importance of spatial effects in real estate literature, this aspect has received limited empirical attention in Croatia. To date, the only comprehensive empirical analysis of spatial spillover effects on Croatian real estate prices is provided by [17], who used a spatial panel model to estimate the impact of tourism, demographic, and economic variables across 556 Croatian cities and municipalities. The variables analyzed included: the share of rental space, the number of overnight stays and tourist arrivals per capita, the share of private accommodation, and demographic indicators (new residents, marriages, vitality index). The increasing share of short-term rentals and private accommodation has the strongest impact on price increases, while the spatial spillover effects are not necessarily limited to immediate neighboring areas, indicating complex interregional dependencies in housing market dynamics.

Although city- and municipality-level analyses provides detailed local insights, a county-level approach can add value by capturing broader regional patterns, increasing the relevance of results for regional policymaking, and providing a more stable and robust statistical framework for analyzing the spatial impact of tourism on real estate prices. For these reasons, the present study focuses on county-level data for Croatia to assess how tourism-related factors contribute to real estate price transmission and spatial spillover effects across regions.

3. Data and methodology

This paper examines the factors influencing real estate prices across the 21 counties of the Republic of Croatia. The analysis covers the period from 2011 to 2019 for each individual county. Although the initial intention was to include data up to 2021, the last two years of the sample (2020 and 2021) were excluded from the sample since they significantly affected the results as they are related to the pandemic period. A spatial panel model is used to assess how selected economic, tourism, and demographic indicators in one county influence real estate prices in neighboring counties, i.e., the counties it borders. The term “spatial” in statistical models refers to spatial econometrics, a method for analyzing interactions between geographical units. In this study, the observed units are the 21 counties of the Republic of Croatia. Spatial dependence poses a methodological challenge for the application of traditional econometric methods, as the assumption of independence and equal distribution of observations is no longer fulfilled. Therefore, it is crucial to use methods that adequately address spatial interactions between geographical units.

Given the structure of the dataset — 21 counties observed over nine years (2011–2019) — a (static) spatial panel model is applied to analyze the relationship between tourism, economic and demographic variables, and residential real estate prices. Table 1 provides a list of variables along with their data sources, while Table 2 presents their descriptive statistics. It should be noted that the selection of variables was guided by the existing literature and the theoretical framework discussed earlier. However, the final selection was largely constrained by the availability of county-level data across the study period.

Variable	Variable description	Source
Real estate price	The median prices of houses and apartments per square meter	MPGI (2024)
Tourist accommodation units	The number of accommodation units in a county	HTZ (2024)
Tourist arrivals (in 000)	The number of tourist arrivals in a county	HTZ (2024)
Share of employees in service activities	Share of employees in service industries within a county compared to the total number of employees in that county (%)	HTZ (2024)
Unemployment rate	Registered unemployment rate (%)	HZZ (2024)
Wages	Net salaries in euros by county	DZS (2024)
Population	Number of inhabitants of the county	DZS (2024)
GDP per capita (EUR)	GDP per capita of a county	DZS (2024)

Table 1: *Definition and source of variables.*

Note: MPGI - Ministry of Physical Planning, Construction and State Assets (Ministarstvo prostornog uređenja, graditeljstva i državne imovine) https://mpgi.gov.hr/; HTZ - Croatian National Tourist Board (Hrvatska turistička zajednica) https://www.htz.hr/hr-HR; HZZ - Croatian Employment Service (Hrvatski zavod za zapošljavanje) https://www.hzz.hr/statistika/; DZS - Croatian Bureau of Statistics (Državni zavod za statistiku) https://podaci.dzs.hr/hr/ [Accessed 24/6/2025]

Significance levels: * $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$.

Variable	Obs	Mean	Std. dev.	Min	Max
Real estate price	189	1006.86	435.36	362	2540
Tourist accommodation units	189	49182.29	80287.15	356	306040
Tourist arrivals (in 000)	189	700.61	1053.23	9	4482
Share of employees in service activities	189	6.16	8.32	0.21	45.32
Unemployment rate	189	18.50	8.60	3.7	38.1
Wages	189	726.15	87.78	581	1100
Population	189	199865.00	164618.70	44625	807254
GDP per capita (EUR)	189	9523.16	2803.13	5285	17959

Table 2: *Descriptive statistics of accepted variables.*

Before applying spatial regression, its application must be justified by establishing the presence of spatial correlation between the geographical units. since the units under investigation are croatian counties, which can be assumed to be interdependent due to shared infrastructure, labor market integration, and regional policy. The expectation of a spatial correlation is further supported by the results of previous studies (see e.g. [17]). To test for cross-sectional dependence, the CD(p) test developed by [11] is employed. This test examines the null hypothesis that assumes the absence of the above dependence. The results of the CD(p) test (Table 3) show that all the analyzed variables are statistically significant, which allows us to reject the null hypothesis and confirm the spatial connectivity between counties. These findings justify the application of spatial econometrics in subsequent analysis.

Variable	CD-test	p-value	corr	abs(corr)
Real estate price	4.96	0.000	0.114	0.387
Tourist accommodation units	26.8	0.000	0.616	0.617
Tourist arrivals (in 000)	33.65	0.000	0.774	0.892
Share of employees in service activities	35.78	0.000	0.823	0.824
Unemployment rate	-1.97	0.048	-0.045	0.322
Wages	-1.98	0.048	-0.045	0.294
Population	24.32	0.000	0.559	0.862
GDP per capita (EUR)	32.12	0.000	0.739	0.847

Table 3: *Test for global dependence of variables.*

Cross-sectional dependence has become a central theme in econometric research, particularly in studies analyzing regional economic dynamics. Traditional panel data models assume independence across cross-sectional units, which is often unrealistic, especially in the case of real estate markets where price transmission frequently occurs between neighboring regions. As a result, the econometric literature increasingly relies on "cross-sectional interaction effects", which indicate that economic behavior in one unit is influenced by behavior in other units. Spatial econometrics extends this approach by explicitly modeling these interactions, especially those driven by geographic proximity. Admittedly, there are alternative approaches for addressing cross-sectional dependence, such as Common Correlated Effects (CCE) estimators. These estimators account for general cross-sectional dependence by modeling correlations arising from unobserved common factors; however, they do not incorporate specific spatial structures. In contrast, spatial panel data models explicitly account for both distance decay and spillover effects across neighboring regions via the spatial weight matrix. This is particularly important in the context of real estate prices, where proximity and regional interdependencies play a crucial role [5].

To further support the use of spatial models, the global Moran's I test is conducted to formally assess the presence of spatial autocorrelation in the data. The test is applied separately for each year, which is appropriate for panel data as it assesses spatial autocorrelation within each cross-section separately. The results (Table 4) indicate that the spatial distribution of the data is not random in most years, with certain periods exhibiting particularly strong spatial clustering. These findings support the use of spatial econometric models for our analysis, as failing to account for spatial dependence could lead to biased results.

Year	Moran's I	Z-score	p-value
2011	0.52457	2.24	0.025
2012	0.14034	0.71	0.48
2013	0.22868	1.02	0.308
2014	0.52678	2.18	0.029
2015	0.62043	2.47	0.014
2016	0.52783	2.22	0.026
2017	0.44717	1.94	0.052
2018	0.44585	1.83	0.068
2019	0.51758	2.07	0.038

Table 4: *Global Moran's I.*

In real estate markets, spatial interactions occur when real estate prices in one county are co-determined by (i) real estate prices in neighboring counties, (ii) explanatory variables such as tourism activity in other counties, and (iii) spatially correlated error terms. These effects, known as "spatial lags", capture the mechanisms through which regional housing markets are interconnected. In contrast to standard panel methods, which do not take such dependencies into account, spatial econometric models — such as the Spatial Autoregressive Model (SAR), the Spatial Durbin Model (SDM) or the Spatial Error Model (SEM) — allow for a more accurate estimation of the impact of tourism on real estate prices in different counties. An important first step in the application of spatial econometrics is the definition and creation of a spatial matrix. A spatial weights matrix is a non-negative N×N matrix where each row represents an observation, while the columns correspond to the locations of neighboring spatial units. Since geographic measures are considered most effective for estimating interactions between spatial units, this study employs a spatial weights matrix based on the connectivity of neighboring counties. We begin by using a row standardized spatial matrix that includes one nearest neighbor (wknn1). To assess the robustness of the results, the same models will be additionally tested using a row-standardized spatial weights matrix based on two nearest neighbors (wknn2), as well as a rook contiguity matrix (wcon), which defines neighbors as counties that share a common border.

To analyze the influence of the independent variables on the dependent variable, three different spatial models—SAR, SDM, and SEM—will be used.

The SAR model focuses on the spatial dependence of the dependent variable. This means it accounts for how the values of the dependent variable in one unit (in this case, a county) depend on the values of the dependent variable in neighboring counties. The SAR model is given as follows:

$$y_t = \rho W y_t + X_t \beta + \mu + \epsilon_t, \quad t = 1, \ldots, T \tag{1}$$

where y_t is a vector of real estate prices for the observed county (i) at year t, while Wy_t represents the spatial lag of the dependent variable, reflecting the influence of real estate prices in neighboring counties. The parameter ρ defines the intensity of the influence of real estate prices in neighboring counties on the prices in the observed county. Additionally, real estate prices in the observed county depend on the county's own vector of variables (X_t) that influence real estate prices, with β indicating the strength of these variables' influence.

The SDM model extends the SAR model by incorporating spatial lags of the independent variables. This model accounts for both the influence of the dependent variable in neighboring counties and the impact of independent variables from neighboring counties on the dependent variable in a given county. It is more flexible because it captures spatial effects on both sides of the equation. The SDM model is described by:

$$y_t = \rho W y_t + X_t \beta + W X_t \theta + \mu + \epsilon_t \tag{2}$$

Compared to the SAR model, this model is extended by $WX_t\theta$. The parameter θ shows the strength of the correlation between the independent variables in neighboring counties and the real estate prices of the observed county.

The SEM model focuses on the spatial dependence of the model errors. It assumes that errors are not independent but have a spatial structure, suggesting that spatial dependence stems from unobserved factors not included in the model. This model is useful when spatial effects influence the data but are not directly measurable. In this analysis, SEM uses a Kapoor-type spatial error structure, which assumes time-varying spatial autocorrelation in the idiosyncratic errors. The SEM model is expressed as:

$$\begin{aligned} y_t &= X_t \beta + \mu + \nu_t \,, \\ \nu_t &= \lambda W \nu_t + \epsilon_t \,. \end{aligned} \tag{3}$$

Here, λ indicates the degree to which errors in one county are related to errors in neighboring counties.

To determine which model best fits the data, a series of diagnostic tests will be conducted. The likelihood ratio (LR) test will evaluate the hypothesis $\theta = -\rho\beta$ to compare SEM and SDM. Rejection of the null hypothesis ($p < 0.05$) favors SEM; otherwise, SDM is preferred. Similarly, SAR and SDM will be compared by testing $\theta = 0$. Rejection of this hypothesis ($p < 0.05$) supports SDM; otherwise, SAR is sufficient.

Additionally, Akaike's Information Criterion (AIC) and Bayesian Information Criterion (BIC) will be used. According to [1], these criteria help analysts select the optimal spatial econometric model and correctly identify spatial dependence. Lower AIC and BIC values indicate better model performance. The results of these diagnostic tests will be reported at the bottom of each results table..

4. Empirical anaylsis

Based on the list of potential determinants (Section 2) and the applied methodology (Section 3), the empirical analysis focuses on examining the impact of tourism and demographic and economic factors on real estate prices in 21 counties of the Republic of Croatia in the period from 2011 to 2019. The analysis is conducted on an annual basis.

First, all variables listed in table 1 are tested, each of which is hypothesized to have a significant impact on real estate prices. For this purpose, the spatial matrix wknn1is used, which considers only one nearest neighbor, i.e., the nearest neighboring county. The full regression results are available in the online appendix at: https://zenodo.org/records/15773699. Diagnostic tests suggest that SAR is the preferred model. Among the tested variables, only three variables are found to be statistically significant in their effects on real estate prices: the number of tourist accommodation units, population, and GDP per capita. The remaining variables are not statistically significant, either when they are all included in the model or when tested in various combinations. Due to space limitations, these results are not presented here but are available upon request.

Given these findings, the analysis is continued using only the three variables and the results are interpreted in more detail. These results are shown in Table 5.

The key component of all spatial models is the coefficient with the spatial matrix, ρ, whose significance indicates the existence of spatial spillover effects across countries. In both Table A1 and Table 5, the spatial ρ is positive and statistically significant, confirming the importance of spatial effects in real estate prices across Croatian counties. In particular, a positive value of spatial ρ indicates that an increase in real estate prices in one county leads to an increase in real estate prices in other counties, i.e., that there are spatial spillover effects.

The interpretation of the coefficients is not as simple as with linear regressions, as spatial regression models also contain information from neighboring counties/observations. In these models, one should therefore distinguish between "direct" and "indirect" effects, while the "overall" average effect of a change in the independent variable on the dependent variable is the combination of these two effects [6]. In our case, the direct effect measures the impact of changes in the underlying factors within a county on the growth of real estate prices in that county when the spatial multiplier is taken into account. The indirect effect measures the impact of changes in the explanatory variables (underlying factors) of other counties on the growth of real estate prices in the county under observation. The main estimation results are shown in the first part of Table 5, while the direct and indirect spillover effects are shown in the lower part of the table.

	(1)	(2)	(3)
Model	**SAR**	**SDM**	**SEM**
Spatial weights matrix	**wknn1**	**wknn1**	**wknn1**
	Real estate price	**Real estate price**	**Real estate price**
Tourist accommodation units	0.00245***	0.00253**	0.00264***
	(0.000817)	(0.00100)	(0.000890)
Population	0.00119***	0.000969**	0.00122***
	(0.000429)	(0.000452)	(0.000475)
GDP per capita	–0.0460**	–0.0248	–0.0482**
	(0.0206)	(0.0240)	(0.0237)
_cons	929.6***	999.7***	1091.7***
	(210.6)	(214.0)	(215.9)
ρ	0.145***	0.131**	
	(0.0554)	(0.0554)	
λ			0.113*
			(0.0626)
LR_ Direct			
Tourist accommodation units	0.00252***	0.00265***	
	(0.000856)	(0.000993)	
Population	0.00119***	0.000998**	
	(0.000421)	(0.000435)	
GDP per capita	–0.0447**	–0.0255	
	(0.0201)	(0.0223)	
LR_ Indirect			
Tourist accommodation units	0.000403*	0.000977	
	(0.000218)	(0.000885)	
Population	0.000186*	0.000629*	
	(0.0000984)	(0.000376)	
GDP per capita	–0.00680*	–0.0424**	
	(0.00387)	(0.0199)	
LR_ Total			
Tourist accommodation units	0.00292***	0.00363***	
	(0.00102)	(0.000997)	
Population	0.00138***	0.00163***	
	(0.000489)	(0.000517)	
GDP per capita	–0.0515**	–0.0679***	
	(0.0230)	(0.0224)	
Number of observations	189	189	189
Number of cross-sectional units	21	21	21
$\theta = -\rho\beta$ **(SEM vs SDM)**		p=0.1015	
$\theta = 0$ **(SDM vs SAR)**	p=0.2843		
AIC	2626.529	2626.906	2629.901
BIC	2649.221	2656.082	2652.594

Table 5: *Main results for SAR, SDM and SEM model mentioned.*

Note: Standard errors are reported in parentheses.
Significance levels: * $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$.

- The hypothesis $\theta = -\rho\beta$ is tested to determine whether the Spatial Durbin Model (SDM) can be simplified to the Spatial Error Model (SEM). The p-value from this test is reported under the model selected based on this criterion.
- The hypothesis $\theta = 0$ is tested to determine whether the Spatial Durbin Model (SDM) can be simplified to the Spatial Autoregressive Model (SAR). The p-value from this test is reported under the model selected based on this criterion.

Since the diagnostic tests suggest the use of SAR model, the results are interpreted using only this model. Both the direct and indirect effects of all independent variables are significant (up to the 10% significance level). The findings can be interpreted as follows.

An increase in the number of tourist accommodation units by 100 is associated with an increase in the average real estate price per square meter in that county of €0.29, of which €0.25 can be attributed to direct effects and €0.04 to indirect effects. For an average 60 m^2 apartment, this corresponds to a total increase of approximately €17.40 in its price. More accommodation units can increase the demand for real estate, as it is expected that tourists, investors and perhaps even local residents will be willing to invest in real estate in that area, which will directly result in an increase in real estate prices. In addition, an increase in tourist accommodation capacity in one county can stimulate tourism development in surrounding areas, increasing the demand for real estate in other neighboring counties, even though the accommodation is not located in those specific areas. The €0.04 per square meter indirect effect reflects a minor but meaningful impact on real estate prices in neighboring counties. For an average 60 m^2 apartment, this indirect effect would lead to a €2.40 increase in its market value, showing that even modest changes in neighboring areas can accumulate and influence broader real estate trends.

An increase in the population of a county by 1,000 is associated with an increase in the price of the average real estate in that county of €1.38 per square meter, of which €1.2 can be attributed to direct and €0.18 to indirect effects. Using the example of an average 60 m^2 apartment, this means an increase in the price of the average apartment of around €82.80, whereby the majority of this increase (€72) is due to direct effects within the county, while the remaining €10.80 is due to indirect effects from neighboring counties. Namely, an increase in population directly contributes to rising real estate prices by increasing demand for housing. As the number of residents grows, so does the pool of potential buyers and tenants. Finally, an increase of the GDP per capita by €100 in a county, leads to a decrease in the price of the average real estate by €5.15, of which €4.47 are direct effects and €0.68 are indirect effects. At first glance, this finding may seem counterintuitive, as one would typically expect that an increase in GDP per capita, i.e., an improvement in the standard of living, would lead to higher real estate prices. However, it is possible that in counties with a higher GDP per capita, i.e., more developed counties, there are more opportunities for investment other than real estate. In such cases, reduced demand for residential property, could result in a decline in housing prices.

To assess the robustness of our results, the spatial weights matrix was changed and the lags of the independent variables were included. These results are given in Table 6.

In Column 1 of Table 6, the two-nearest neighbors weight matrix (wknn1) is used, while in Column 2, the contiguity matrix is applied. The results are largely consistent across both specifications, with the only notable difference being that GDP per capita becomes statistically insignificant in one specification. In Column 3, the current values of the dependent variables are replaced with their spatial lags to address potential endogeneity concerns. As in the main specification, the results remain robust, confirming the validity of our findings. Additionally, the spatial coefficient, rho, is statistically significant in all model specifications.

	(1)	(2)	(3)
Model	**SAR**	**SAR**	**SAR**
Spatial weights matrix	**wknn2**	**wcon**	**wknn1**
	Real estate price	**Real estate price**	**Real estate price**
Tourist accommodation units	0.00238***	0.00226***	
	(0.000823)	(0.000699)	
Population	0.00115***	0.00102***	
	(0.000434)	(0.000364)	
GDP per capita	−0.0452**	−0.0253	
	(0.0210)	(0.0193)	
Lag_Tourist accommodation units			0.00219***
			(0.000726)
Lag_Population			0.000966**
			(0.000387)
Lag_GDP per capita			−0.0161
			(0.0240)
_cons	901.5***	508.5**	653.1***
	(221.0)	(221.8)	(210.6)
ρ	0.189**	0.427***	0.180***
	(0.0764)	(0.102)	(0.0549)
LR_Direct			
Tourist accommodation units	0.00246***	0.00250***	
	(0.000862)	(0.000784)	
Population	0.00115***	0.00109***	
	(0.000426)	(0.000381)	
GDP per capita	−0.0438**	−0.0256	
	(0.0205)	(0.0203)	
Lag_Tourist accommodation units			0.00227***
			(0.000767)
Lag_Population			0.000973**
			(0.000383)
Lag_GDP per capita			−0.0142
			(0.0239)
LR_Indirect			
Tourist accommodation units	0.000558*	0.00164*	
	(0.000329)	(0.000845)	
Population	0.000253*	0.000706*	
	(0.000144)	(0.000372)	
GDP per capita	−0.00937	−0.0152	
	(0.00573)	(0.0139)	
Lag_Tourist accommodation units			0.000449**
			(0.000220)
Lag_Population			0.000191*
			(0.000101)
Lag_GDP per capita			−0.00277
			(0.00500)
LR_Total			
Tourist accommodation units	0.00302***	0.00414***	
	(0.00110)	(0.00147)	
Population	0.00140***	0.00180***	
	(0.000523)	(0.000682)	
GDP per capita	−0.0532**	−0.0408	
	(0.0246)	(0.0331)	
Lag_Tourist accommodation units			0.00272***
			(0.000943)
Lag_Population			0.00116**
			(0.000464)
Lag_GDP per capita			−0.0170
			(0.0286)
N	189	189	168
N_g	21	21	21

Table 6: *Robustness checks.*

Note: Standard errors in parentheses.
* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$.

5. Conclusion

This paper analyzes the impact of tourism, economic and demographic factors on real estate prices across all 21 counties of the Republic of Croatia, with a particular focus on spatial spillover effects between counties. Covering the period from 2011 to 2019, the study offers a novel perspective by using counties as the primary geographical units of analysis — unlike prior research in Croatia, which has predominantly focused on cities, municipalities, or the national level. The aim of this research was to show that changes in real estate prices in a county are not only the result of local factors, but also reflect interdependencies with neighboring counties.

The results confirm the existence of spatial dependence in all 21 counties and show significant spillover effects in real estate prices across them. Key factors influencing real estate prices include the number of accommodation units, population size and GDP per capita. Although additional factors such as tourist arrivals, the share of employees in the service sector and the unemployment rate were tested, their effects were not found to be statistically significant.

Importantly, the results highlight the critical role of spatial panel models in capturing these spillover effects between counties. By accounting for spatial dependence, a better understanding is gained of how changes in one county can trigger chain reactions in neighboring counties, leading to fluctuations in real estate prices. This underscores the need to use econometric models that consider spatial dimensions to fully capture the dynamics of real estate markets. The results confirm the significant influence of neighboring factors, emphasizing that any real estate policy design must take these spatial relationships into consideration.

The contribution of this study lies in the fact that it accounts for spatial interdependencies in real estate markets at the county level offering a better understanding of price dynamics compared to traditional models. By explicitly modeling these spatial effects, this study contributes to filling a gap in the literature and provides valuable insights for the design of county-level policy, particularly in the context of tourism-driven real estate markets.

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Risk-based decision-making: extended F-Entropy method with correction factor based 2D-FAHP method

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Abstract. In complex decision-making processes using multi-criteria decision-making methods, experts' beliefs and judgments, as well as their knowledge, are influential in the comparisons between criteria and alternatives, which may lead to overlapping importance rankings of criteria. In this study, to increase the degree of representativeness of uncertainties in overlapping criteria evaluations, a correction term is integrated into the fuzzy entropy method, so that the risk cost levels of the main criteria of equal importance differ according to the number of sub-criteria. The integrated correction term into the fuzzy entropy calculation is proposed as a relative importance weight multiplier in the two-dimensional fuzzy AHP process steps. Although the α-alpha truncation method is proposed for the Fuzzy AHP method, the decision matrix is converted into crisp values in the process stages and directly reduced to the Classical AHP method. Within this article, using the judgment matrix with interval values is proposed instead of exact judgment values as to not distort the fuzzy structure of the result values. In the article, the two-dimensional fuzzy Entropy weights obtained by using the alpha-cutting method can be used as a relative importance weight multiplier in the fuzzy AHP process proposed in two dimensions to obtain clear importance rankings of the criteria. Thanks to the method approach proposed, OCTAVE Allegro logic is combined to create a relative risk matrix according to the risk environment conditions for the criteria alongside the ranking level.

Keywords: Fuzzy entropy, Fuzzy AHP, Decision-Making, Two-Dimensional FAHP

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1. Introduction

Digital information assets are the data, documents, and systems in electronic media that organizations create, store, and manage to support decision-making processes and provide competitive advantage. These assets are exposed to various risks such as unauthorized access, disruption of data integrity, and cyber threats. Since traditional measures such as firewalls, antivirus programs and encryption are insufficient, dynamic decision-making processes including multi-criteria evaluation and risk-based approaches are needed [21]. As the need for dynamic and risk-based decision-making processes for the security of digital information assets is increasing, the basis of these processes is the identification and analysis of threats. Risk-based decision-making approaches for the protection of digital information assets identify potential threats, systematically analyze their likelihood and impact, and enable effective risk management by

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ensuring that limited resources are allocated to the most critical security measures. In this context, identifying threats, examining the organizational structure, analyzing and prioritizing human interactions, relationships between subsystems and security attitudes contribute to more effective determination of security strategies. In particular, methods such as OCTAVE etc., which address the degree of risk, the probability and impact of risk factors, allow organizations to systematically assess and manage risks and create proactive decision mechanisms by continuously monitoring risk management [6]. Within this article, the risk levels of critical information assets under unstable environmental conditions will be determined within the framework of protection of digital information assets and efficient use of resources. The aim is to conduct more specific research on the use of Analytic Hierarchy Process (AHP) and entropy methods together in two-dimensional (2D) analysis and to contribute to the enrichment of literature.

Considering the time and financial constraints, the high level of risk structure and uncertainty in ensuring the confidentiality-integrity-availability of information requires the use of multi-criteria decision-making methods in establishing the framework of information security risk assessment. Classical multi-criteria decision-making methods used in decision-making theory, which are based on personal judgments alongside the handling of risks, uncertainties and complex-valued problems over a period of time, can be transformed into a flexible structure with a fuzzy logic approach because they contain imprecise information, and it is difficult to assign performance levels to criteria. Fuzzy logic approaches system behavior is used where analytical functions are unavailable. In cases where the behavior is not well understood, solutions involving complex systems can be obtained that provide fast and approximate solutions. In determining the appropriate solution structure, experts must select from a set of alternatives. In the decision process for selecting alternatives, many factors such as organizational needs, goals, risks, benefits, resources, etc. are considered. Accordingly, a systematic and objective evaluation process is needed for decision makers to determine the most appropriate alternative, and multi-criteria decision-making techniques help determine the most appropriate option among the alternatives.

The AHP method, developed for solving complex multi-criteria problems, ranks the decision options from high to low importance within the scope of the criteria determined by the decision maker(s) among multiple options [17]. AHP is based on the assumption that subjective judgments cluster decision elements according to their common characteristics and includes the preferences, knowledge and intuition of groups or individuals in the decision-making process [20]. In the decision-making process, the more information the criterion attribute provides to the decision maker, the more effective that attribute is in the decision-making process. In this context, the real weight of the criteria attributes, which are the source of information in decision-making, includes both objective and subjective judgments simultaneously. At this point, while AHP is one of the methods that commonly use subjective judgments in multi-criteria decision-making processes, the Entropy method provides objective weighting. The Entropy method, which determines criterion weights, is based on the information value and discrimination of the criteria and contributes to obtaining more rational and consistent results by providing an objective weighting in the decision process [2]. In analyzing the complexity of decision-making processes, hybrid models created by integrating multi-criteria decision-making methods with different methods increase the accuracy and efficiency of the decision-making process. Feizi et al. [7] showed that a hybrid weighting method created by integrating TOPSIS, a multi-criteria decision-making method, with AHP and Shannon Entropy improves the decision-making processes in spatial analysis. Kumar [14] performed risk assessment using a fuzzy-based analytical hierarchy process. Gundogdu et al. [8] stated that by integrating the Illustrated Fuzzy AHP and Linear Assignment Model, more precise and reliable results are obtained in decision processes involving uncertainty. Duleba et al. [5] used the interval-valued global fuzzy AHP method to provide a more effective evaluation of uncertain data. Navdeep and Dixit [1] performed spatial risk assessment by integrating AHP and fuzzy AHP models with the Shannon Entropy and frequency ratio method. Nasrullah et al. [16] proposed the use of RIPC4 and the AHP in the risk assessment of e-government to analyze risk priorities. Kaur et al. [12] identified and prioritized risks in decentralized finance using the fuzzy analytical hierarchy process (FAHP). Hybrid model studies using Extended Fuzzy Entropy and 2D FAHP together are almost non-existent in the literature. Keleş [13] evaluates the trade facilitation performance of E7 countries by integrating multiple weighting and ranking methods.

When reviewing the literature, there is no theoretical proposition regarding the weighting of a large number of main criteria that are considered to be of equal importance and have different and large numbers of sub-criteria. This leads to misleading results in determining the riskiness levels of the criteria and revealing the cost levels of this riskiness. In the process steps used in decision-making methods, even if there are different numbers of sub-criteria, the aforementioned problems persist since they are weighted with equal importance. Therefore, in order to solve this problem, the Yaşar-Terzioğlu approach is proposed in this paper, which takes into account the number of sub-criteria when applying weighting methods. In this approach, in order to increase the degree of representativeness of uncertainties in conflicting criteria evaluations, a correction term $\sqrt{\frac{T-t_i}{T-1}}$ that takes into account the differences in the number of sub-criteria is integrated into the fuzzy entropy method, so that the risk cost levels of the main criteria of equal importance differ according to the number of sub-criteria. Therefore, the application of existing hybrid models differs from the studies in literature with the addition of the correction factor proposed in the paper. In addition, interval values are needed to create the risk pool for the riskiness levels. At the end of the solution phase of the fuzzy AHP method, ranking the criteria based on the crisp value does not make it possible to form the risk pool. For this reason, in this article, it is shown that the fuzzy AHP method can be combined with the α-alpha cut-off method to ensure the formation of the risk pool, and that the solution can be obtained based on the two-dimensional confidence interval instead of the crisp value until the last stage of the solution. Thus, unlike the hybrid models in literature, the criteria rankings are analyzed as intervals instead of crisp values. As a result, the proposed approach differs from other hybrid models in literature, firstly, by applying the criteria weight calculation steps presented in the paper and secondly, by maintaining the fuzzy structure until the end of the process and considering the process steps for creating a risk pool.

Hybrid model studies using Extended Fuzzy Entropy and 2D Fuzzy Analytic Hierarchy Process (AHP) together are almost non-existent in the literature. However, by including a risk response stage and a classification such as risk acceptance/mitigation/deferral, an approach that does not only focus on measuring and ranking risk is also presented in the literature. In this article, the Extended F-Entropy Method with Correction Factor- Based 2D-FAHP Method is presented, and an application example is given.

2. Methodology

While evaluations can be made intuitively when a single criterion is considered in the decision-making processes, evaluations become complex when more than one criterion is involved. If the main criteria in the decision matrices based on expert opinions are considered equally important even though they have sub-criteria of different dimensions, no definite judgment can be made in their importance ranking. Therefore, the AHP method is not an appropriate method in cases where there is uncertainty due to the size of the comparisons and the complexity of the calculations. Since the same decision matrix is used in the fuzzy AHP method, which is a transformation of the AHP method in modeling uncertainty, it cannot provide a successful solution in providing a clear ranking even if the fuzzy structure is switched. In addition, since the clarification process step used to reach the solution in the fuzzy AHP method converts the decision matrix into a definite judgment matrix and reduces it to a crisp value, the fuzzy structure disappears, and the relative risk matrix cannot be created.

To achieve a crisp importance ranking for overlapping criteria, this paper proposes preserving the fuzzy structure by presenting the results as a range, rather than reducing them to integer values. The alpha cut method reconstructs the fuzzy decision matrix within the confidence interval boundaries. In the following stage, each criterion's two-dimensional fuzzy entropy (F-entropy) weights must be calculated using the number of sub-criteria as a multiplier. Finally, by multiplying the weight obtained from the two-dimensional F-entropy method with the two-dimensional decision matrix generated using the alpha value, a two-dimensional fuzzy AHP (2D-FAHP) solution can be obtained to obtain the criteria's crisp ranking

When dealing with complex decision elements, the AHP method, which is based on the assumption of clustering decision elements based on common characteristics, creates a hierarchical structure with the goal at the top level. At the next level, criteria and, if any, subcriteria affecting the goal are identified, and a structure of alternatives determining the outcome is developed. The pairwise comparison matrix, based on the judgments of the decision maker, is constructed using fuzzy triangular numbers, where d_{ij}^k represents the k-th decision maker's preference of the i-th criterion over the j-th criterion. The triangular number dimension is defined as $a_{ij}^k = (a_1, a_2, a_3)$, resulting in the fuzzy decision matrix $\tilde{A}^k = (\tilde{a}_{nn}^k)$. The confidence interval of the triangular values obtained by comparing the fuzzy comparison values of each criterion is defined at the α-alpha level as $l_{ij} = (\tilde{a}_2 - \tilde{a}_1)\alpha + \tilde{a}_1$ and $u_{ij} = \tilde{a}_3 - (\tilde{a}_3 - \tilde{a}_2)\alpha$ for $0 \leq \alpha < 1$. The triangular fuzzy number is characterized as $A_\alpha = [\tilde{a}_1^\alpha, \tilde{a}_3^\alpha] = [l_{ij}, u_{ij}]$, thus reaching the two-dimensional fuzzy confidence interval values for each criterion [11]. For $\alpha = 1$, the values of l_{ij} and u_{ij} are the same as the triangular fuzzy values. Moreover, because no confidence interval is formed, the range for the α-cut value is proposed as $0 \leq \alpha < 1$.

In MCDM problems with uncertain judgments, fuzzy and interval numbers are used to weight criteria with F-entropy based on the α-cut method [10]. The decision matrix for the problem is constructed in the following manner:

$$D = \begin{array}{c} \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{i1} & x_{i2} & \cdots & x_{in} \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix} \\ \tilde{w}_1 \;\; \tilde{w}_2 \;\; \cdots \;\; \tilde{w}_n \end{array} \tag{1}$$

where x_{ij} represents the value of the i-th alternative $(i = 1, 2, \ldots, m)$ with respect to the j-th evaluation criterion $(j = 1, 2, \ldots, n)$, and $\tilde{w}_j$, which represents the weight of the j-th criterion [15]. When $(\tilde{x}_{ij})_\alpha = \{x \in \mathbb{R} \mid \mu_{\tilde{x}_{ij}}(x) \geq \alpha\}$, the α-cut set is represented in the form of an interval number as:

$$[\tilde{x}_{ij}^L(\alpha), \tilde{x}_{ij}^U(\alpha)] = \left[\min\left\{x_{ij} \mid x_{ij} \in \mathbb{R} \text{ and } \mu_{\tilde{x}_{ij}}(x_{ij}) \geq \alpha\right\},\right.$$
$$\left.\max\left\{x_{ij} \mid x_{ij} \in \mathbb{R} \text{ and } \mu_{\tilde{x}_{ij}}(x_{ij}) \geq \alpha\right\}\right],$$

where $0 < \alpha \leq 1$. In the first step of the Entropy method,the following formula are used to create a normalized decision matrix to normalize criteria with different units to the range $[0, 1]$:

$$p_{ij}^l = \frac{x_{ij}^l}{\sum_{i=1}^n x_{ij}^u}, \quad p_{ij}^u = \frac{x_{ij}^u}{\sum_{i=1}^n x_{ij}^u} \tag{2}$$

The entropy (uncertainty measure) values for the evaluation criterion are calculated as:

$$e_j^l = \min\left\{-k \sum_{i=1}^m p_{ij}^l \ln p_{ij}^l,\; k \sum_{i=1}^n p_{ij}^u \ln p_{ij}^u\right\}$$

$$e_j^u = \max\left\{-k\sum_{i=1}^{m} p_{ij}^l \ln p_{ij}^l,\; k\sum_{i=1}^{n} p_{ij}^u \ln p_{ij}^u\right\} \tag{3}$$

where $k = (\ln(m))^{-1}$ and $0 \leq e_j \leq 1$.In the next step, after calculating the degrees of differentiation of the information expressed as $d_i^l = 1 - e_{ij}^u$ and $d_i^u = 1 - e_{ij}^l$, the weight values for the criteria are obtained by satisfying the condition $\sum_{j=1}^{n} w_j = 1$, as follows [15];

$$w_i^l = \frac{d_j^l}{\sum_{j=1}^{n} d_j^u} \qquad w_i^u = \frac{d_j^u}{\sum_{j=1}^{n} d_j^l} \tag{4}$$

This paper proposes employing the two-dimensional decision matrix from the FAHP method using the α-cut method instead of the decision matrix defined in Equation (1) of the F-entropy method. In this context, for the value of the j-th criterion (for $j = 1, 2, \ldots, n$) with respect to the i-th criterion (for $i = 1, 2, \ldots, n$), the element $x_{ij} = (l_{ij}, u_{ij})$ represents a two-dimensional fuzzy number. Thus, the two-dimensional fuzzy decision matrix is constructed in the following manner:

$$\tilde{A} = [x_{ij}]_{n\times n} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} \tag{5}$$

where $x_{ij} = (l_{ij}, u_{ij})$ denotes the two-dimensional fuzzy relative importance value of the j-th criterion with respect to the i-th criterion, and l_{ij} and u_{ij} are expressed as the lower and upper bounds, respectively, of the relative importance value x_{ij}^{α}. To calculate the values in the normalized decision matrix for each criterion, we first obtained the vector sum of each x_i. When calculating the power of (-1) of the total vector, the lower (l_{ij}) and upper (u_{ij}) values are swapped, transforming the triangular number-based decision matrix into a two-dimensional fuzzy number form.

In addition to the normalization process in Equation (2), which employs Shannon's [19] extended F-entropy method, this study recommends incorporating subcriteria ratios when the relevant main criteria of equal importance have different numbers of subcriteria. Shannon Entropy is an approach that forms the basis of other entropy methods and is preferred as the most convenient method because it minimizes the computational burden. In this context, In this context, this paper proposed the Yaşar-Terzioğlu approach that uses the weight multiplier $\sqrt{\frac{T-t_i}{T-1}}$ for the sub-criteria if the ratio of the number of the i-th criterion subcriteria, t_i, to the total number of sub-criteria, T, is greater than 5%. The normalization process for calculating the fuzzy weight value for the i-th criterion, which includes the added weight multiplier for correction, should be carried out in two dimensions using the following formulations:

$$p_{ij}^l = \frac{x_{ij}^l}{\sum_{i=1}^{n} x_{ij}^u} \times \sqrt{\frac{T-t_i}{T-1}}, \quad p_{ij}^u = \frac{x_{ij}^u}{\sum_{i=1}^{n} x_{ij}^l} \times \sqrt{\frac{T-t_i}{T-1}} \tag{6}$$

where the calculations are performed in two dimensions. Equation (3) defines the lower and upper bounds for the F-entropy (uncertainty measure) values related to the evaluation criterion for $j = 1, 2, \ldots, n$:

$$e_j^l = -k\sum_{i=1}^{m} p_{ij}^l \ln p_{ij}^l, \quad e_j^u = -k\sum_{i=1}^{m} p_{ij}^u \ln p_{ij}^u \tag{7}$$

where the two-dimensional lower and upper bounds are calculated. The F-Entropy weights for the criteria are proposed to be obtained using the formulation:

$$w_j^l = \frac{1 - e_j^u}{\sum_{j=1}^n d_j^l + \sum_{j=1}^n d_j^u}, \quad w_j^u = \frac{1 - e_j^l}{\sum_{j=1}^n d_j^l + \sum_{j=1}^n d_j^u} \tag{8}$$

instead of the formula given in Equation (4), where $\sum_{j=1}^n d_j^l = \sum_{j=1}^n (1 - e_j^l)$ and $\sum_{j=1}^n d_j^u = \sum_{j=1}^n (1 - e_j^u)$.

The paper's proposed 2D-FAHP method uses the α-cut method to reduce the two-dimensional fuzzy decision matrix to two dimensions. It incorporates the two-dimensional F-Entropy weights w_j^l and w_j^u as row multipliers for the main criteria, as suggested in the paper.

The FAHP comparison values weighted according to the two-dimensional F-Entropy method should be calculated as two-dimensional values for each criterion, where $\tilde{X} = [x_{ij} \cdot w_j^{(l,u)}]_{(n \times n)} = [\tilde{x}_{ij}]_{(n \times n)}$, with $\tilde{x}_{ij} = (x_{ij}^l \cdot w_j^l, x_{ij}^u \cdot w_j^u) = (\tilde{x}_{ij}^l, \tilde{x}_{ij}^u)$. In the 2D-FAHP framework, the normalization process should be performed using the following formulation, incorporating the relevant criterion's F-entropy weight multiplier into the row operations:

$$\tilde{a}_{ij}^l = \frac{\tilde{x}_{ij}^l}{\sum_{i=1}^n \tilde{x}_{ij}^u} \quad \text{and} \quad \tilde{a}_{ij}^u = \frac{\tilde{x}_{ij}^u}{\sum_{i=1}^n \tilde{x}_{ij}^l} \tag{9}$$

The geometric mean of the two-dimensional criteria weights' lower and upper bound values should be used for the 2D-FAHP method proposed in this paper, rather than the arithmetic mean suggested by Buckley [3]. To obtain the two-dimensional criterion weights, we use the following formula:

$$w_i^{(l,u)} = \begin{cases} w_i^l = \left(\prod_{j=1}^n \tilde{a}_{ij}^l\right)^{\frac{1}{n}} \\ w_i^u = \left(\prod_{j=1}^n \tilde{a}_{ij}^u\right)^{\frac{1}{n}} \end{cases} \tag{10}$$

In the paper context, Hurwicz's realism criterion is included as a smoothing process to allow the decision-maker to incorporate their emotions into the process [9]. Using the fuzzy two-dimensional weights obtained from Equation (10), the fuzzy membership function is defined with the risk index β as:

$$G_i = \left((w_i^{(l,u)})^\alpha\right)^\beta = \beta \cdot w_i^l + (1 - \beta) \cdot w_i^u, \quad 0 \leq \beta \leq 1, \ 0 \leq \alpha \leq 1, \ i < j \tag{11}$$

The risk index β is a measure of the degree of risk in an environment: the values of 0, 0.5, and 1 indicate a high-risk (pessimistic), moderate-risk (moderate), and low-risk (optimistic) environments, respectively [17].

Probability	**30–45**	**16–29**	**0–15**
High	Pool 1	Pool 2	Pool 2
Medium	Pool 2	Pool 2	Pool 3
Low	Pool 3	Pool 3	Pool 4

Pools	**Risk Mitigation Approach**	**Pools**	**Risk Mitigation Approach**
Pool 1	Reduce the Risk	Pool 3	Delay or accept risk
Pool 2	Reduce or delay risk	Pool 4	Accept the risk

Table 1: *Relative Risk Matrix*

Moreover, an approach is suggested in this paper to streamline the risk assessment process by incorporating Hurwicz's realism criterion into the OCTAVE Allegro method and converting it into a risk matrix (Table 1). Scores are computed for each environmental condition to quantify the criteria. Risk score ranges are established, and risk pools are generated by ranking the criteria with the highest scores based on the riskiest environment. Risks earning the highest scores are Pool 1; the next highest score range, Pool 2; risks within the next highest score range are assigned to Pool 3; and risks within the lowest score range are assigned to Pool 4. Distinct risk pools are established for criteria and alternatives, with each pool allocated to evenly distributed segments based on their relative risk score and probability of occurrence [4].

3. Application

The main and sub-criteria for selecting risky assets in the scope of this paper are based on the Digital Transformation Office's expert opinions, accessible open-source guides, and related literature reviews. This article considers the AHP method by Saaty [18] as the structure used in the selection decisions of the riskiest criteria addressing the security risk of digital information assets. For the AHP method, the hierarchical structure of the Security Risks for Digital Information Assets, which includes the main criteria and sub-criteria, is grouped (Table 2). The hierarchical structure obtained in Table 2 is based on the Trakya University Scientific Research Project No. 2021/133 "E-Government Information Security Risk Assessment: Artificial Neural Network Modeling." Pairwise comparison matrices were created with the data used in the hierarchical structure grouping.

Networks & Systems	**Application & Data Security**	**Portable Device & Environment Security**
Hardware Inventory	File/Resource Security	Smartphone/Tablet
Software Inventory	External Integrations	Laptop Security
Threat/Vulnerability Mgmt.	Installation/Config.	Media Security
Email Security	Secure Dev.	Comm. Security
Malware Protection	Logging Mgmt.	
Network Security	Authentication	
DLP	Malicious Prevention	
Monitoring	Session Auth. Mgmt.	
Virtualization	DB Management	
Incident Mgmt.	Authorization	
Pen. Tests		
Access Mgmt.		
BC/DR Mgmt.		
Remote Work		
IoT & Device Security	**Physical Security**	**Personnel Security**
Network Services	General Precautions	General Precautions
Internal Storage	System Room Security	Awareness Activities
Auth./Access	TEMPEST Protection	Supplier Relations
API Security		
Other Measures		

Table 2: *E-Government Information Security Risk Structure*

At the stage of constructing the pairwise comparison matrices, complementings the subjective judgments of the experts, the pairwise comparison matrix of the main criteria and the normalized comparison matrix are obtained in Table 3. In the pairwise comparison matrix, where the relative importance of the criteria against each other is determined, the values in each column are normalized by dividing by the sum of the relevant columns, and then the weight vector of the criteria is obtained by averaging each row. When the importance weights of the criteria are analyzed, the Application & Data Security and Portable Device & Environment criteria have the same importance ranking.

Since the pairwise comparison matrix is consistent and the consistency ratio is less than 0.10 in Table 3, the FAHP method based on the Buckley approach is used to eliminate the

Pairwise Comparison Matrix						
	N&S	A & DS	PD & E	IoT	SPP	SP
Networks & Systems	1,00	0,33	0,33	4,00	2,00	4,00
Application & Data Security	3,00	1,00	1,00	6,00	5,00	4,00
Portable Device & Environment	3,00	1,00	1,00	6,00	5,00	4,00
Internet of Things	0,25	0,17	0,17	1,00	0,50	0,33
Security of Physical Places	0,50	0,20	0,20	2,00	1,00	3,00
Security of Personnel	0,25	0,25	0,25	3,00	0,33	1,00
Normalized Pairwise Comparison Matrix						
	N&S	A & DS	PD & E	IoT	SPP	SP
Networks & Systems	0,13	0,11	0,11	0,18	0,14	0,24
Application & Data Security	0,38	0,34	0,34	0,27	0,36	0,24
Portable Device & Environment	0,38	0,34	0,34	0,27	0,36	0,24
Internet of Things	0,03	0,06	0,06	0,05	0,04	0,02
Security of Physical Places	0,06	0,07	0,07	0,09	0,07	0,18
Security of Personnel	0,03	0,08	0,08	0,14	0,02	0,06
Weight Vector						
Networks & Systems: 0,153				Internet of Things: 0,041		
Application & Data Security: 0,322				Security of Physical Places: 0,090		
Portable Device & Environment: 0,322				Security of Personnel: 0,070		
Consistency Evaluation of the Corresponding Matrix						
$\lambda_{\max} = 6{,}4$	Consistency Index = 0,07			Consistency Ratio = 0,0585		

Table 3: *AHP Pairwise Comparison Matrix and Consistency Evaluation of Main Criteria*

judgment uncertainty in the criteria with equal importance ranking. FAHP better manages uncertain and imprecise data. The evaluations of decision makers are expressed using fuzzy numbers instead of integers, and thus the uncertainty of subjective judgments is modeled. In Table 4, the AHP decision matrix is reconstructed according to the triangular fuzzy numbers and a fuzzy pairwise comparison matrix is obtained. After the relevant normalization process steps, the fuzzy triangular weights of the criteria are obtained, and the Application & Data Security and Portable Device & Environment criteria have the same importance ranking.

Fuzzy Pairwise Comparison Matrix						
	N&S	A & DS	PD & E	IoT	SPP	SP
Networks & Systems	(1,1,1)	(1/4,1/3,1/2)	(1/4,1/3,1/2)	(3,4,5)	(1,2,3)	(3,4,5)
Application & Data Security		(1,1,1)	(1,1,1)	(5,6,7)	(4,5,6)	(3,4,5)
Portable Device & Environment			(1,1,1)	(5,6,7)	(4,5,6)	(3,4,5)
Internet of Things				(1,1,1)	(1/3,1/2,1)	(1/4,1/3,1/2)
Security of Physical Places					(1,1,1)	(1,1,1)
Security of Personnel						(1,1,1)
Fuzzy Triangular Weights						
Networks & Systems	0,00033475		0,00493767		0,07811667	
Application & Data Security	0,07141402		0,49993887		3,49962672	
Portable Device & Environment	0,07141402		0,49993887		3,49962672	
Internet of Things	0,00000020		0,00000161		0,00002777	
Security of Physical Places	0,00000551		0,00005555		0,00078117	
Security of Personnel	0,00000952		0,00006510		0,00061722	
Defined Values and Importance Ranking						
Networks & Systems	0,027796	3	Internet of Things	0,000010	6	
Application & Data Security	1,356993	1*	Security of Physical Places	0,000281	4	
Portable Device & Environment	1,356993	1*	Security of Personnel	0,00231	5	

Table 4: *Fuzzy Pairwise Comparison Matrix and Evaluation Results*

When the AHP results (Table 3) and FAHP results (Table 4) are analyzed, the importance rankings of the Application & Data Security and Portable Device & Environment criteria still overlap. To determine the continuity of the overlap in the importance ranking at different levels of uncertainty, the alpha-cutting method is examined using the fuzzy pairwise comparison matrix in Table 4. In Table 5, the Hurwicz coefficient and alpha-intercept values are used together to obtain the confidence levels of the evaluations of the decision makers and the net importance rankings by including the uncertain environmental conditions in the method.

Main Criteria						
Hurwicz Coefficient	$\beta = 0$					
Alpha-Cut Values	**0**	**0,2**	**0,4**	**0,6**	**0,8**	**1,0**
Networks & Systems	0,19	0,18	0,18	0,17	0,16	0,15
Application & Data Security	0,32	0,32	0,32	0,32	0,32	0,32
Portable Device & Environment	0,32	0,32	0,32	0,32	0,32	0,32
Internet of Things	0,04	0,04	0,04	0,04	0,04	0,04
Security of Physical Places	0,05	0,05	0,06	0,06	0,06	0,07
Security of Personnel	0,06	0,06	0,06	0,07	0,07	0,07
Hurwicz Coefficient	$\beta = 0{,}5$					
Alpha-Cut Values	**0**	**0,2**	**0,4**	**0,6**	**0,8**	**1,0**
Networks & Systems	0,16	0,16	0,16	0,16	0,16	0,15
Application & Data Security	0,32	0,32	0,32	0,32	0,32	0,32
Portable Device & Environment	0,32	0,32	0,32	0,32	0,32	0,32
Internet of Things	0,04	0,04	0,04	0,04	0,04	0,04
Security of Physical Places	0,06	0,06	0,06	0,07	0,07	0,07
Security of Personnel	0,07	0,07	0,07	0,07	0,07	0,07
Hurwicz Coefficient	$\beta = 1$					
Alpha-Cut Values	**0**	**0,2**	**0,4**	**0,6**	**0,8**	**1,0**
Networks & Systems	0,13	0,14	0,14	0,15	0,16	0,16
Application & Data Security	0,31	0,31	0,31	0,31	**0,31***	0,32
Portable Device & Environment	0,31	0,31	0,31	0,31	**0,32***	0,32
Internet of Things	0,04	0,04	0,04	0,04	0,04	0,04
Security of Physical Places	0,09	0,08	0,08	0,07	0,07	0,07
Security of Personnel	0,10	0,09	0,09	0,08	0,08	0,07

Table 5: *General Weights for Hurwicz Coefficient (β) and Alpha-Cut Values (α)*

As shown in Table 5, the Application & Data Security and Portable Device & Environment criteria have the same importance ranking even if all uncertain risky situations are included in the process. Only in the 0.8 alpha-intercept uncertainty environment under β=1, which refers to the risk-free (optimistic) environment conditions, the importance ranking changes. For the other conditions, the overlap in the importance ranking continues. As a result, when Tables 3–5 are considered, it is determined that the overlap in the importance rankings of the Application & Data Security and Portable Device & Environment criteria continues and the approaches in the literature cannot eliminate this overlap, especially in risky and medium-risk environments.

This paper criticizes using a fuzzy structure to obtain the importance ranking and then clarifying this structure and reducing it to crisp values. Instead, it is more effective to perform the importance ranking directly within the fuzzy structure and compare interval values instead of crisp values. Accordingly, the Hurwicz coefficients given in Table 5 should not be used to return to the crisp value and instead be evaluated directly by the alpha-intercept method. Thus, when obtaining the importance ranking in the fuzzy model approaches, the uncertainty will be reflected as a range instead of a point. Typically, solving a system that starts with a fuzzy structure by reducing it to a single value will not give realistic results. For this reason, the following steps should be applied in order.

In Table 6, the AHP decision matrix is first transformed into a fuzzy triangular decision matrix to address fuzziness. Within the scope of the method proposed, the α-alpha cut-off method is used to calculate the lower and upper bound values of the fuzzy triangular values for the 0.8 environment level, which indicates that the uncertainty is high (other alpha cut-off values can also be examined if desired by the researchers) and the pairwise comparison matrix is obtained in two dimensions.

AHP Decision Matrix						
	Net. & Sys	App. & Sec	Port. Dev	IoT	Phys. Sec	Pers. Sec
Net. & Sys	1,00	0,33	0,33	4,00	2,00	4,00
App. & Sec	3,00	1,00	1,00	6,00	5,00	4,00
Port. Dev	3,00	1,00	1,00	6,00	5,00	4,00
IoT	0,25	0,17	0,17	1,00	0,50	0,33
Phys. Sec	0,50	0,20	0,20	2,00	1,00	3,00
Pers. Sec	0,25	0,25	0,25	3,00	0,33	1,00
Fuzzy Triangular Decision Matrix						
	Net. & Sys	App. & Sec	Port. Dev	IoT	Phys. Sec	Pers. Sec
Net. & Sys	(1,1,1)	(0.25,0.33,0.5)	(0.25,0.33,0.5)	(3,4,5)	(1,2,3)	(3,4,5)
App. & Sec	(2,3,4)	(1,1,1)	(1,1,1)	(5,6,7)	(4,5,6)	(3,4,5)
Port. Dev	(2,3,4)	(1,1,1)	(1,1,1)	(5,6,7)	(4,5,6)	(3,4,5)
IoT	(0.20,0.25,0.33)	(0.14,0.17,0.20)	(0.14,0.17,0.20)	(1,1,1)	(0.33,0.5,1)	(0.25,0.33,0.5)
Phys. Sec	(0.33,0.50,1)	(0.17,0.20,0.25)	(0.17,0.20,0.25)	(1,2,3)	(1,1,1)	(1,1,1)
Pers. Sec	(0.20,0.25,0.33)	(0.20,0.25,0.33)	(0.20,0.25,0.33)	(2,3,4)	(1,1,1)	(1,1,1)
Two-Dimensional Reduced Decision Matrix with Alpha-Cut Method						
α=0.8	Net. & Sys	App. & Sec	Port. Dev	IoT	Phys. Sec	Pers. Sec
Net. & Sys	(1,1)	**(0.32, 0.37)**	(0.32,0.37)	(3.80, 4.20)	(1.80, 2.20)	(3.80, 4.20)
App. & Sec	(2.80, 3.20)	(1,1)	(1,1)	(5.80, 6.20)	(4.80, 5.20)	(3.80, 4.20)
Port. Dev	(2.80, 3.20)	(1,1)	(1,1)	(5.80, 6.20)	(4.80, 5.20)	(3.80, 4.20)
IoT	(0.24,0.27)	(0.16, 0.17)	(0.16,0.17)	(1,1)	(0.47,0.60)	(0.32, 0.37)
Phys. Sec	(0.47, 0.60)	(0.19,0.21)	(0.19, 0.21)	(1.80, 2.20)	(1,1)	(1,1)
Pers. Sec	(0.24, 0.27)	(0.24, 0.27)	(0.24, 0.27)	(2.80, 3.20)	(1,1)	(1,1)
Total	7.55; 8.53	2.91; 3.02	2.91; 3.02	21.00; 23.00	13.87; 15.20	13.72; 14.97

Comparison of ***Network & System*** *with* ***Application & Data Security*** *(0.25;0.33;0.50) for* $\alpha = 0.8$*: Two-Dimensional Values for Comparison of Network & System with Application & Data Security:*

$Networks\&Systems_{l_{12}} = [0.8 \times (0.33 - 0.25)] + 0.25 = 0.314 \approx 0.32$

$Networks\&Systems_{u_{12}} = 0.50 - [(0.50 - 0.33) \times 0.8] = 0.364 \approx 0.37$

Table 6: *Calculation of the Two-Dimensional Decision Matrix with the α-Cut Method*

The normalization of the 2D decision matrix obtained by the alpha-cutting method in Table 6 is performed by integrating the correction term $\sqrt{\frac{T-t_i}{T-1}}$ proposed in Equation (6) into the fuzzy entropy method in Table 7. Additionally, as proposed in Equations (7) and (8), the 2D F-entropy values ($e_j^{(l,u)}$) and weights ($w_j^{(l,u)}$) for the main criteria are also obtained in Table 7. The 2D F-entropy importance weights, obtained as suggested in Table 6, should be used as the relative importance multiplier of the criteria in the 2-dimensional FAHP method. Thus, the importance values of the overlapping criteria are differentiated.

Normalized Decision Matrix Using the Extended F-Entropy Method with a Correction Factor						
$\alpha = 0.8$	Net. & Sys	App. & Sec	Port. Dev	IoT	Phys. Sec.	Pers. Sec.
Net. & Sys.	(0.09,0.10)	**(0.09, 0.11)**	(0.10,0.12)	(0.15, 0.18)	(0.11, 0.15)	(0.24, 0.29)
App. & Sec.	(2.80, 3.20)	(1,1)	(1,1)	(5.80, 6.20)	(4.80, 5.20)	(3.80, 4.20)
Port. Dev	(2.80, 3.20)	(1,1)	(1,1)	(5.80, 6.20)	(4.80, 5.20)	(3.80, 4.20)
IoT	(0.24,0.27)	(0.16, 0.17)	(0.16,0.17)	(1,1)	(0.47,0.60)	(0.32, 0.37)
Phys. Sec	(0.47, 0.60)	(0.19,0.21)	(0.19, 0.21)	(1.80, 2.20)	(1,1)	(1,1)
Pers. Sec	(0.24, 0.27)	(0.24, 0.27)	(0.24, 0.27)	(2.80, 3.20)	(1,1)	(1,1)
Sub-Criteria Ratios	14/39	10/39	4/39	5/39	3/39	3/39
Correction Factor Weight Multiplier	0.81	0.87	0.96	0.95	0.97	0.97

Comparison of ***Network & System*** *with* ***Application & Data Security*** *Normalized Weights*

$p^l_{ij} = \frac{x^l_{ij}}{\sum_{i=1}^n x^u_{ij}} \times \sqrt{\frac{T-t_i}{T-1}}, \quad p^u_{ij} = \frac{x^u_{ij}}{\sum_{i=1}^n x^l_{ij}} \times \sqrt{\frac{T-t_i}{T-1}}$

$\text{Networks\&Systems}^l_{l_{12}} = \left(\frac{0,32}{3,02}\right) \times \sqrt{\frac{39-10}{39-1}} = 0{,}092$

$\text{Networks\&Systems}^u_{l_{12}} = \left(\frac{0,37}{2,91}\right) \times \sqrt{\frac{39-10}{39-1}} = 0{,}110$

Two-Dimensional Fuzzy Entropy Method Decision Matrix Using the Number of Sub-Criteria						
$\alpha = 0.8$	Net. & Sys	App. & Sec	Port. Dev	IoT	Phys. Sec.	Pers. Sec.
Net. & Sys.	(-0.22, -0.24)	**(-0.21, -0.24)**	(-0.23, -0.25)	(-0.29, -0.31)	(-0.24, -0.28)	(-0.34, -0.36)
App. & Sec.	(-0.35, -0.36)	(-0.35, -0.36)	(-0.36, -0.36)	(-0.34, -0.35)	(-0.36, -0.36)	(-0.34, -0.36)
Port. Dev.	(-0.35, -0.36)	(-0.35, -0.36)	(-0.36, -0.36)	(-0.34, -0.35)	(-0.36, -0.36)	(-0.34, -0.36)
IoT	(-0.08, -0.10)	(-0.14, -0.15)	(-0.15, -0.16)	(-0.13, -0.14)	(-0.10, -0.13)	(-0.08, -0.09)
Phys. Sec.	(-0.13, -0.17)	(-0.16, -0.17)	(-0.17, -0.18)	(-0.19, -0.22)	(-0.17, -0.18)	(-0.17, -0.18)
Pers. Sec.	(-0.08, -0.10)	(-0.18, -0.20)	(-0.19, -0.21)	(-0.24, -0.27)	(-0.17, -0.18)	(-0.17, -0.18)

Comparison of ***Network & System*** *with* ***Application & Data Security*** *Two dimensional Fuzzy Entropy values*

$e^l_j = -k \sum_{j=1}^m p^l_{ij} \ln p^l_{ij}, \quad e^u_j = -k \sum_{i=1}^m p^u_{ij} \ln p^u_{ij}$

$\text{Networks\&Systems}_{l_{12}} = 0{,}092 \times \ln(0{,}092) = -0{,}219$

$\text{Networks\&Systems}_{u_{12}} = 0{,}110 \times \ln(0{,}110) = -0{,}243$

$\text{Networks\&Systems } e^l_{12} = [(-0{,}219) + (-0{,}359) + (-0{,}359) + (-0{,}143) + (-0{,}161) + (-0{,}185)] \times (-0{,}5581) = 0{,}80$

$\text{Networks\&Systems } e^u_{12} = [(-0{,}243) + (-0{,}361) + (-0{,}361) + (-0{,}154) + (-0{,}174) + (-0{,}202)] \times (-0{,}5581) = 0{,}83$

Calculation of Two-Dimensional Fuzzy Entropy Criterion Weights						
1/ln=0.5581	**Networks & Systems**	**Application & Data Security**	**Portable Device & Environment**	**Internet of Things**	**Security of Physical Places**	**Security of Personnel**
$e_j^{(l,u)}$	(0.69,0.76)	**(0.80, 0.83)**	(0.83, 0.86)	(0.86, 0.94)	(0.80, 0.85)	(0.82, 0.87)
$d_j^{(l,u)}$ = 1- $e_j^{(l,u)}$	(0.31, 0.24)	**(0.20, 0.17)**	(0.17, 0.14)	(0.14, 0.06)	(0.20, 0.15)	(0.18, 0.13)
$\sum_{j=1}^n d^l_j + \sum_{j=1}^n d^u_j$	2,08					
Two-Dimensional Fuzzy Entropy Weights	(0.117,0.148)	**(0.079, 0.097)**	(0.065, 0.083)	(0.031, 0.065)	(0.070, 0.096)	(0.064, 0.085)

Comparison of ***Network & System*** *with* ***Application & Data Security*** *Two dimensional Fuzzy Entropy weights*

$\left(w^l_j = \frac{1-e^u_j}{\sum_{j=1}^n d^l_j + \sum_{j=1}^n d^u_j}, \quad w^u_j = \frac{1-e^l_j}{\sum_{j=1}^n d^l_j + \sum_{j=1}^n d^u_j}\right)$

Networks & Systems: $w^l_1 = \frac{0.17}{2.08} \approx 0.079, \quad w^u_1 = \frac{0.20}{2.08} \approx 0.097$

$\Rightarrow w_1^{(l,u)} = (0.079,\ 0.097)$

(Two-Dimensional Entropy Weights for the comparison of the first criterion 'Networks & Systems' with the second criterion 'Application & Data Security')

Table 7: *Calculation of Two-Dimensional Fuzzy Entropy Weights*

In Table 8, the relative importance ($w_j^{(l,u)}$) weights obtained by the F-entropy method, in which the correction term proposed in the article is integrated, create a difference in the importance rankings of the overlapping criteria in the 2D fuzzy AHP method. The weights w_i^l and w_i^u of the fuzzy 2-dimensional entropy are applied to the fuzzy decision matrix, reduced to two dimensions by the alpha-cutting method, as row multipliers of the relevant main criteria, as proposed. By adding the fuzzy entropy relative importance multiplier of the relevant criterion to the row operations, the normalization process is applied using Equation (9) within the scope of the 2D FAHP. In the calculation of the weights of the criteria, the geometric mean is used to reduce the effect of extreme values between the lower and upper limits. Using geometric means, a 2-dimensional prioritization (weight) vector is obtained, and the importance rankings are made according to the lower and upper bound ranges. In this way, there is no need for any reduction to the net value as a result of any calibration process.

Two-Dimensional FAHP Decision Matrix						
$\alpha = 0.8$	Net. & Sys.	App. & Sec.	Port. Dev.	IoT	Phys. Sec.	Pers. Sec.
Net. & Sys.	(1,1)	**(0.32, 0.37)**	(0.32, 0.37)	(3.80, 4.20)	(1.80, 2.20)	(3.80, 4.20)
App. & Sec.	(2.80, 3.20)	(1, 1)	(1, 1)	(5.80, 6.20)	(4.80, 5.20)	(3.80, 4.20)
Port. Dev.	(2.80, 3.20)	(1, 1)	(1, 1)	(5.80, 6.20)	(4.80, 5.20)	(3.80, 4.20)
IoT	(0.24, 0.27)	(0.16, 0.17)	(0.16, 0.17)	(1, 1)	(0.47, 0.60)	(0.32, 0.37)
Phys. Sec.	(0.47, 0.60)	(0.19, 0.21)	(0.19, 0.21)	(1.80, 2.20)	(1, 1)	(1, 1)
Pers. Sec.	(0.24, 0.27)	(0.24, 0.27)	(0.24, 0.27)	(2.80, 3.20)	(1, 1)	(1, 1)
Total	(7.55, 8.53)	**(2.91, 3.02)**	(2.91, 3.02)	(21.0, 23.0)	(13.87, 15.20)	(13.72, 14.97)
Calculation of 2D Fuzzy AHP Criterion Weights						
$\alpha = 0.8$	Net. & Sys.	App. & Sec.	Port. Dev.	IoT	Phys. Sec.	Pers. Sec.
2D Fuzzy Entropy Weights	**(0.117, 0.148)**	(0.079, 0.097)	(0.065, 0.083)	(0.031, 0.065)	(0.070, 0.096)	(0.064, 0.085)
Net. & Sys.	(0.01, 0.02)	**(0.012, 0.019)**	(0.012, 0.019)	(0.019, 0.030)	(0.014, 0.023)	(0.030, 0.045)
App. & Sec.	(0.026, 0.041)	(0.026, 0.033)	(0.026, 0.033)	(0.020, 0.029)	(0.025, 0.036)	(0.020, 0.020)
Port. Dev.	(0.021, 0.035)	(0.021, 0.031)	(0.021, 0.031)	(0.016, 0.025)	(0.020, 0.031)	(0.016, 0.028)
IoT	(0.001, 0.002)	(0.002, 0.004)	(0.002, 0.004)	(0.001, 0.003)	(0.001, 0.003)	(0.001, 0.002)
Phys. Sec.	(0.004, 0.008)	(0.004, 0.007)	(0.004, 0.007)	(0.005, 0.010)	(0.005, 0.007)	(0.005, 0.007)
Pers. Sec.	(0.002, 0.003)	(0.005, 0.008)	(0.005, 0.008)	(0.008, 0.013)	(0.004, 0.006)	(0.004, 0.006)
Priority Vector and Ranking						
	Net. & Sys.	App. & Sec.	Port. Dev.	IoT	Phys. Sec.	Pers. Sec.
Priority Vector	**(0.016, 0.024)**	(0.024, 0.031)	(0.019, 0.026)	(0.004, 0.003)	(0.005, 0.007)	(0.004, 0.006)
Rank	3	1	2	6	4	5

$\tilde{x}_{ij} = (x_{ij}^l \cdot w_j^l,\ x_{ij}^u \cdot w_j^u) = (\tilde{x}_{ij}^l,\ \tilde{x}_{ij}^u)$, where $i = 1, 2, \ldots, n$

$$\tilde{a}_{ij}^l = \frac{\tilde{x}_{ij}^l}{\sum_{i=1}^{n} x_{ij}^u} \quad ; \quad a_{ij}^u = \frac{\tilde{x}_{ij}^u}{\sum_{i=1}^{n} x_{ij}^l}$$

Example calculation for a_{12}:

$$\tilde{a}_{12}^l = (0.32 \cdot 0.117) \cdot \frac{1}{3.02} = 0.012 \quad ; \quad a_{12}^u = (0.37 \cdot 0.148) \cdot \frac{1}{2.91} = 0.019$$

The Two-Dimensional Fuzzy AHP Weight for the Comparison Between the First Criterion, 'Network and System,' and the Second Criterion, 'Application & Data Security':

$(\tilde{a}_{12}^l,\ a_{12}^u) = (0.012,\ 0.019)$

Fuzzy weight calculation:

$$\left(w_i^l = \left(\prod_{j=1}^{n} \tilde{a}_{ij}^l \right)^{1/n} \quad ; \quad w_i^u = \left(\prod_{j=1}^{n} \tilde{a}_{ij}^u \right)^{1/n} \right) = (0.016,\ 0.024)$$

Table 8: *Two-Dimensional Weighted Fuzzy AHP*

Within the article, it is concluded that the method applied on the digital information assets system is effective in decomposing overlapping criteria. Through the proposed process steps, it is revealed that the Application & Data Security and Portable Device & Environment criteria have different importance rankings. In multi-criteria decision problems with multiple main criteria with different number of sub-criteria, the selection process of risky digital information assets can become complicated for decision makers. Especially when there is no distinction between criteria that appear to have the same importance, organizations have difficulty in deciding which criterion to develop a prevention strategy or make a financial investment for. The proposed method determines the degree of importance between the criteria more precisely and ensures that each criterion has different weights. Thus, decision makers are provided with a more rational and prioritized evaluation opportunity and strategic planning processes become more effective. Additionally, the paper proposes to integrate the Hurwicz criterion into the OCTAVE Allegro method and transform it into a relative risk matrix to facilitate the risk assessment process. In Tables 9 and 10, the risk score of the criteria for each environmental condition is calculated and the risk score ranges are determined, and risk pools are created by ranking the importance of the criteria with high scores according to the riskiest environment. To integrate the decision maker's feelings according to the decision process into the importance values in the prioritization vector, a relative risk matrix is created by grouping the criteria according to different risk environments using Equation (11).

	$\beta = 0$	$\beta = 0{,}5$	$\beta = 1$
Networks & Systems	0,244	0,201	0,159
Application & Data Security	0,315	0,276	0,237
Portable Device & Environment	0,295	0,244	0,194
Internet of Things	0,029	0,020	0,011
Security of Physical Places	0,075	0,060	0,045
Security of Personnel	0,067	0,055	0,043

Calculation of Adjusted Fuzzy Weights Using α-Cut and β-Weighting

$G_i = \beta \cdot w_i^l + (1-\beta) \cdot w_i^u, \quad 0 \leq \beta \leq 1, \quad 0 \leq \alpha \leq 1, \quad i < j$

Given: $\alpha = 0.8$

Criterion: Network & Systems

$\beta = 0 : (0 \times 0.016) + (1-0) \times 0.024 = 0.24$

$\beta = 0{,}5 : (0.5 \times 0.016) + (1-0.5) \times 0.024 = 0.20$

$\beta = 1 : (1 \times 0.016) + (1-1) \times 0.024 = 0.16$

Table 9: *The Risk Matrix for Two-Dimensional Weighted Fuzzy AHP*

	0,38–0,26	0,25–0,13	0,12–0,00
$\beta = 0$	1. Application & Data Security, 2. Portable Device & Environment	3. Networks & Systems	4. Security of Physical Places, 5. Security of Personnel, 6. Internet of Things
$\beta = 0{,}5$	1. Application & Data Security	2. Portable Device & Environment, 3. Networks & Systems	4. Security of Physical Places, 5. Security of Personnel, 6. Internet of Things
$\beta = 1$		1. Application & Data Security, 2. Portable Device & Environment, 3. Networks & Systems	4. Security of Physical Places, 5. Security of Personnel, 6. Internet of Things

Table 10: *The Hurwicz Risk Matrix*

4. Conclusion

In multicriteria decision problems, the AHP method is used to determine criteria weights through pairwise comparisons of alternatives and criteria, and to compute and rank the alternative's relative importance. However, when dealing with uncertainty and overlapping criterion evaluations, the AHP method is insufficient. Therefore, the FAHP method is used to perform more precise evaluations under conditions of indecisiveness and uncertainty. FAHP, which expresses and calculates using triangular numbers, produces more precise, adaptable, and practical results than the classical AHP method. However, neither method is completely adequate for evaluating overlapping criterion sets. This is because both methods focus on directly converting fuzzy structures into crisp values, introducing uncertainty into the results and risk structure. Although the α-cut method is suggested in fuzzy approaches to increase the degree of crisp values representing fuzzy numbers, the decision matrix is still converted to crisp values during the processing stages, thus reducing the FAHP method back to the classical AHP method and preventing the attainment of crisp importance rankings for overlapping criteria. In this article, providing an effective method for the selection of risky criteria is based on creating a hierarchical structure by using the AHP method to ensure the consistency of the decision matrix and eliminating the fuzziness in expert opinions with FAHP. However, in case the fuzziness is insufficient in separating the conflicting criteria, it is suggested that the FAHP method is first combined with the α - alpha cut-off method to create a two-dimensional confidence interval of expert opinions. This two-dimensional confidence interval aims to provide a healthier balance between different expert opinions. The normalization process in the two-dimensional F-entropy method obtained by the alpha cut method is carried out, and also proposed to include the sub-criteria ratios as a correction factor in the calculation stages if the main criteria have different sub-criteria. In this way, the main criteria alongside the sub-criteria are effectively included in the evaluation. The integration of the correction factor into the fuzzy entropy method, which is proposed to be used in this paper, aims to eliminate the imbalances between the weights of the criteria. Thus, a more balanced risk distribution is ensured, while concurrently, criteria with different numbers of sub-criteria are prevented from being of equal importance to each other and a healthier prioritization opportunity is provided to decision makers. This approach helps to evaluate the criteria of different dimensions and scope in a balanced manner. In particular, it becomes easier for organizations to direct their limited resources to the most critical areas and risk management processes become more effective. Within the scope of this paper, it is proposed to use the two-dimensional F-entropy importance weights obtained by integrating the correction factor as the relative importance multiplier of the criteria in the two-dimensional FAHP method. In this context, overlapping criteria can be separated and the importance values of the relevant main criteria can be differentiated, allowing more precise determination of the determined importance values. The two-dimensional values of the differentiated and non-overlapping criteria were grouped according to different environmental conditions, and a risk pool was created for the criteria. The method proposed in this paper can increase the efficiency and accuracy of the evaluation process and can be an effective tool in achieving the desired goals.

Attribution

This study is derived from the implementation of the doctoral thesis currently being conducted by Aysu Yaşar at the Institute of Social Sciences, Trakya University, under the supervision of Prof. Dr. M. Kenan Terzioğlu.

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Multidecision criteria models for Logistics Performance Index in the EU countries

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Abstract. Key factors influencing multicriteria logistics performance in the European Union (EU) include the weighting of evaluation criteria, infrastructure efficiency, logistics quality, and environmental impact. The Logistics Performance Index (LPI), which evaluates logistics systems across six dimensions, highlights how the importance assigned to these criteria significantly impacts national rankings. This study aims to enhance the understanding of logistics performance dynamics within Europe, supporting informed decision-making and continuous improvement in the region's logistics sector. To determine country rankings, the study applies the CRiteria Importance Through Intercriteria Correlation (CRITIC) method for weighting criteria, alongside the Measurement of Alternatives and Ranking according to COmpromise Solution (MARCOS), Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS), and Simple Additive Weighting (SAW) methods. The analysis utilizes LPI indicators from the 2023 World Bank dataset. The results indicate that Finland maintains its leading position in the LPI ranking. However, a significant number of countries experience shifts in their rankings depending on the weight assigned to each criterion, underscoring the sensitivity of logistics performance evaluation to methodological choices.

Keywords: CRITIC, Logistics Performance Index, MARCOS, SAW, TOPSIS

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1. Introduction

Logistics performance in the European Union (EU) is influenced by several key factors, primarily assessed through the Logistics Performance Index (LPI). The LPI, developed by the World Bank, is a comprehensive benchmarking tool that provides a numerical score for each country based on factors such as infrastructure quality, customs efficiency, logistics competence, tracking and tracing capabilities, and the timeliness of shipments. These dimensions are essential for evaluating logistics systems and for guiding strategic decisions in transportation and trade.

This index evaluates logistics performance across multiple criteria, and the relative weighting of these criteria can significantly affect the ranking of EU countries. Understanding the dynamics and sensitivity of these rankings is crucial for enhancing logistics efficiency and competitiveness at both national and regional levels.

In recent years, a growing body of literature has examined the limitations of the LPIs equal weighting approach and emphasized the value of multi-criteria decision-making (MCDM) models for more nuanced assessments [4, 10, 21]. MCDM methods enable decision-makers to evaluate multiple, often conflicting, factors simultaneously, offering a more adaptable and context-sensitive framework for logistics analysis. Various techniques have been explored, including CRITIC, AHP, and Entropy for weighting, and methods such as TOPSIS, MARCOS, and SAW for ranking [3, 5, 6].

Moreover, new approaches such as the Green Logistics Performance Index (GLPI) have highlighted the increasing importance of environmental sustainability in logistics benchmarking, especially in light of the EU's ambitious CO_2 free urban logistics targets by 2030 [17]. These developments underscore the need for methodological transparency and multi-method validation in assessing national logistics systems [15, 19].

The contribution of this paper lies in its integrated application of several MCDM methods - CRITIC for objective criteria weighting, and MARCOS, TOPSIS, and SAW for alternative ranking - to evaluate the logistics performance of EU countries using the 2023 World Bank LPI dataset. While prior studies have applied individual methods, few have systematically compared multiple MCDM techniques on the same dataset. By doing so, this study not only strengthens the robustness of the results but also provides insights into how methodological choices influence country rankings.

This research supports informed decision-making in logistics strategy and policy by identifying performance disparities among EU countries, highlighting the methodological sensitivity of LPI rankings, and offering a replicable framework for similar evaluations in other regional contexts.

2. Literature review

The LPI provides information on various aspects of logistics performance, including customs efficiency, infrastructure quality, ease of arranging shipments, and timeliness. Countries are ranked based on their scores, which help identify strengths and weaknesses in their logistics systems. According to [1], LPI is essential for policy makers and stakeholders who want to improve logistics performance at the national and regional levels.

Infrastructure quality is a critical determinant of logistics performance. Research indicates that without adequate infrastructure, logistics services cannot achieve optimal efficiency. The integration of infrastructure efficiency with logistics quality and environmental impact is vital for sustainable logistics performance, as evidenced by benchmark studies in EU countries [8].

The environmental impact of logistics operations is increasingly recognized as a significant factor. Many EU countries struggle to balance operational efficiency with environmental sustainability, indicating the need for policies that address both aspects simultaneously [8]. In contrast, while the focus on logistics performance is essential, some argue that an overemphasis on ranking may overlook the unique challenges faced by individual countries, potentially leading to misaligned policy priorities.

The LPI is calculated using six equally weighted criteria, but the importance of these weights can vary significantly [4]. Studies have shown that adjusting these weights can alter the ranking of countries, highlighting the need for tailored evaluations based on specific national contexts [11]. The LPI reveals that the importance assigned to these criteria significantly affects national rankings, as demonstrated by various multi-criterion decision making (MCDM) methods employed in recent studies [5, 11, 22].

The integration of subjective and objective weighting methods improves the robustness of logistics performance evaluations, allowing countries to strategically focus on areas for improvement [21]. In general, these factors collectively shape the logistics landscape within the EU, influencing both regional and global competitiveness [4].

Methods such as CRITIC and FUCOM have been used to determine the optimal weighting of these criteria, demonstrating that sensitivity analysis is essential for accurate evaluations [4, 6, 21]. The CRITIC method is a robust objective weighting technique that balances variability and correlation to ensure criteria independence. However, alternative methods like Entropy (objective), AHP and BWM (subjective), or SWARA (semi-subjective) can be used based on data availability and decision-maker preferences. Selecting the appropriate method depends on whether the study emphasizes statistical objectivity (CRITIC, Entropy) or expert-driven evaluation (AHP, BWM, SWARA) [3].

Comparing the applied method to rank the different countries, MARCOS method provides precise and robust rankings but is complex and not yet widely validated. TOPSIS is easy to use and widely applied but suffers from sensitivity to normalization and rank reversal issues. SAW is computationally simple and stable but lacks the sophistication needed for handling more complex decision-making scenarios. The choice of method depends on the specific requirements of the decision problem, such as the complexity of alternatives, sensitivity to criteria weights, and the need for ideal solution considerations. [3]. These models enable organizations to make informed decisions that align with improving their LPI scores.

3. Methodology

The research starts by identifying pertinent studies from academic databases that align with the subject. It consolidates key insights from prior research on logistics performance indicators in EU nations, focusing on commonly utilized indicators, benchmarking standards, methodological best practices, and empirical findings regarding the determinants of logistics efficiency and effectiveness in the region.

Subsequently, logistics performance indicators (LPIs) were gathered, and potential quantitative methods were examined to address the distinct challenges and complexities associated with evaluating logistics efficiency and effectiveness in this context. Figure 1 illustrates the analytical approach adopted for assessing these indicators.

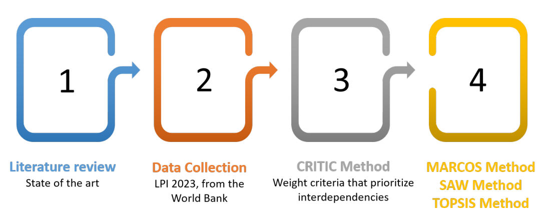


Figure 1: *Evaluation framework of logistics performance*

3.1. Data collection and spatial analysis

The World Bank gathers LPI data through a combination of surveys, interviews, and expert evaluations conducted with logistics professionals, policymakers, and business leaders in each country. These survey tools are designed to capture both qualitative and quantitative aspects of logistics performance, employing standardized questionnaires and scoring frameworks.

The collected LPI data are publicly available through multiple online platforms [24]. While the LPI covers countries worldwide, this study specifically focuses on European Union nations, with an emphasis on data from the 2023 report.

Geographic visualization of these criteria offers valuable insights into spatial disparities and regional logistics performance. By mapping these indicators, it becomes possible to better understand the contribution of each factor to overall logistics efficiency. Additionally, this approach supports evidence-based policymaking by identifying regional strengths and areas requiring improvement, ensuring that strategic decisions are informed by both quantitative rankings and their spatial distribution.

MARCOS, SAW, and TOPSIS were selected due to their complementary strengths in handling both additive and compromise-based decision logic, as well as their established use in logistics and performance benchmarking contexts.

3.2. Decision matrix and decision process

Let's consider a decision matrix in a usual multi-criteria decision making problem [14]. Each value of x_{ij} represents the achievement by the country i in the criterium j.

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix}. \tag{1}$$

In a multi-decision criteria context, it is important to define if the criterium have the goal of maximizing (max) or minimizing (min) decision-making process [3]; in this case, all the six criteria are benefit, i.e., it should be maximized.

The goal is to evaluate and rank the alternatives (countries) according to the specified indicators. This assessment is performed by assigning weights w_j (where $w_j \geq 0$ and $\sum w_j = 1$) to different criteria. Then, the value for each country (V_i) should be calculated using weight additive function:

$$V_i = \sum_{j=1}^{n} w_j x_{ij} \tag{2}$$

3.3. CRITIC Method

The weights described in the previous subsection have different ways to be calculated. One of them is the CRITIC method. The CRITIC (Criteria Importance Through Inter-criteria Correlation) method offers a nuanced perspective by considering the inter-dependencies among decision criteria [10]. Logistics performance should consider interdependencies, as they affect the entire process of sending orders between countries. The CRITIC method comprises the four sequential steps:

Step 1. Define the decsion matrix according to the alternatives and the criteria.

Step 2. Normalize the initial decision matrix D according to the criteria type:

$$\text{benefit crit.:} \quad y_{ij} = \frac{x_{ij} - \min_i x_{ij}}{\max_i x_{ij} - \min_i x_{ij}}; \qquad \text{cost crit.:} \quad y_{ij} = \frac{x_{ij} - \max_i x_{ij}}{\min_i x_{ij} - \max_i x_{ij}} \tag{3}$$

Step 3. Calculate the symmetric linear correlation matrix $[r_{ij}]$, using statistical measures such as Pearson correlation coefficient. These correlations form the foundation for assigning relative weights to the criteria, where stronger correlations indicate greater importance.

Step 4. Calculate the objective weights as

$$w_j = \frac{K_j}{\sum_{j=1}^{n} K_j}, \tag{4}$$

where $K_j = \sigma_j \sum_{j=1}^{n}(1 - r_{ij})$, with σ_j corresponding to the standard deviation of the each column of $[y_{ij}]$.

After the weight process has been concluded, it is necessary to rank the countries to multi-criteria techniques. In this paper there are used three different techniques for this assessment: MARCOS, SAW and TOPSIS.

3.4. MARCOS Method

The MARCOS (Measurement of Alternatives and Ranking according to COmpromise Solution) method integrates correspondence analysis and similarity matrices to rank alternatives based on their performance across multiple criteria [18, 23]. This approach is goal-oriented and data-driven, ranking alternatives based on their similarity and removing the need for subjective weighting of criteria [16].

Step 1. After defining the decision matrix, it is necessary to extend it by adding two new rows for each criterion: the ideal alternative (AI) and the anti-ideal alternative (AAI), as follows:

$$\begin{array}{ll} \text{benefit crit.:} & AAI = min_i x_{ij} \text{ and } AI = max_i x_{ij} \\ \text{cost crit.:} & AAI = max_i x_{ij} \text{ and } AI = min_i x_{ij} \end{array} \tag{5}$$

Step 2. The normalization process for this method is:

$$\text{benefit criterion:} \quad n_{ij} = \frac{x_{AI}}{x_{ij}} \qquad \text{cost criterion:} \quad n_{ij} = \frac{x_{ij}}{x_{AI}} \tag{6}$$

Step 3. The determination of the weighted matrix ($[v_{ij}]$), with the weights (w_j) calculated in the previous subsection, is done as:

$$v_{ij} = n_{ij} \times w_j \tag{7}$$

Step 4. The determination of the utility degree of alternatives K_i can be calculated as:

$$K_i^- = \frac{S_i}{S_{AAI}} \qquad K_i^+ = \frac{S_i}{S_{AI}} \tag{8}$$

where S_i represents the sum of the elements of the weighted matrix $[v_{ij}]$, and S_{AI} and S_{AAI} corresponds to the sum of elements of alternative ideal and anti-ideal, respectively.

Step 5. The determination of the utility function of alternatives $f(K_i)$:

$$f(K_i) = \frac{K_i^+ + K_i^-}{1 + \frac{1 - f(K_i^+)}{f(K_i^+)} + \frac{1 - f(K_i^-)}{f(K_i^-)}}, \quad \text{with} \quad f(K_i^-) = \frac{K_i^+}{K_i^+ + K_i^-}; \; f(K_i^+) = \frac{K_i^-}{K_i^+ + K_i^-} \tag{9}$$

Step 6. The rank of the alternatives (countries) with the best values of the utility functions.

3.5. SAW Method

The Simple Additive Weighting (SAW) Method is a well-known approach within the family of multicriteria decision-making (MCDM) methods. It is widely applied in fields such as operations research, engineering, environmental science, and economics to address problems involving multiple criteria that must be balanced to make a decision. The SAW method is particularly popular due to its simplicity, intuitiveness, and effectiveness in aggregating diverse criteria into a single score for ranking alternatives [7, 20].

The SAW method is based on the principle of weighted linear combination, where each alternative's performance across criteria is aggregated into a single score. The alternative with the highest score is typically considered the most preferred [13].

Step 1. After defining the decision matrix, it is necessary to normalize the matrix, using the following criterion type:

$$\text{benefit criterion:} \quad n_{ij} = \frac{x_{ij}}{max(x_{ij})} \qquad \text{cost criterion:} \quad n_{ij} = \frac{min(x_{ij})}{x_{ij}} \tag{10}$$

Step 2. The determination of the weighted matrix ($[v_{ij}]$), with the weights (w_j) calculated in the previous subsection, is done as:

$$v_{ij} = n_{ij} \times w_j \tag{11}$$

Step 3. For each alternative i, calculate its total score S_i as:

$$S_i = \sum_{j=1}^{n} v_{ij} \tag{12}$$

Step 4. The rank of the alternatives on their total scores S_i is the one with the highest score is the best option.

3.6. TOPSIS Method

The Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) is a widely used multi-criteria decision-making (MCDM) method that ranks alternatives based on their closeness to ideal and anti-ideal solutions. TOPSIS can handle both quantitative and qualitative criteria, making it applicable in various fields, including manufacturing process optimization [2, 9]. The method's popularity stems from its simplicity, rationality, and computational efficiency [12].

Step 1. After defining the decision matrix, it is necessary to normalize the matrix, using the following criterion type:

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}}, \quad \forall i, j \tag{13}$$

Step 2. The determination of the weighted matrix ($[v_{ij}]$), with the weights (w_j) calculated in the previous subsection, is done as:

$$v_{ij} = n_{ij} \times w_j \tag{14}$$

Step 3. Determine the Ideal and Anti-Ideal solutions, as follows:

- Ideal Solution (A^+) is the best performance for each criterion:

$$A^+ = \{v_1^+, v_2^+, \dots, v_n^+\}, \quad v_j^+ = \begin{cases} \max(v_{ij}), & \text{if } j \text{ is a benefit criterion} \\ \min(v_{ij}), & \text{if } j \text{ is a cost criterion} \end{cases} \tag{15}$$

- Anti-Ideal Solution (A^-)is the worst performance for each criterion:

$$A^- = \{v_1^-, v_2^-, \dots, v_n^-\}, \quad v_j^- = \begin{cases} \min(v_{ij}), & \text{if } j \text{ is a benefit criterion} \\ \max(v_{ij}), & \text{if } j \text{ is a cost criterion} \end{cases} \tag{16}$$

Step 4. Calculate the Separation Measures, using the distance of each alternative from the Ideal and Anti-Ideal solutions:

- Distance from Ideal Solution (S_i^+):

$$S_i^+ = \sqrt{\sum_{j=1}^{n} (v_{ij} - v_j^+)^2}, \quad \forall i \tag{17}$$

- Distance from Negative-Ideal Solution (S_i^-):

$$S_i^- = \sqrt{\sum_{j=1}^{n} (v_{ij} - v_j^-)^2}, \quad \forall i \tag{18}$$

Step 5. Compute the Relative Closeness to the Ideal Solution, calculating the relative closeness C_i of each alternative to the ideal solution:

$$C_i = \frac{S_i^-}{S_i^+ + S_i^-}, \quad \forall i \tag{19}$$

The value C_i lies between 0 and 1, where a value closer to 1 indicates a better alternative.

Step 6. Rank the alternatives in descending order of C_i. The alternative with the highest C_i is considered the best choice.

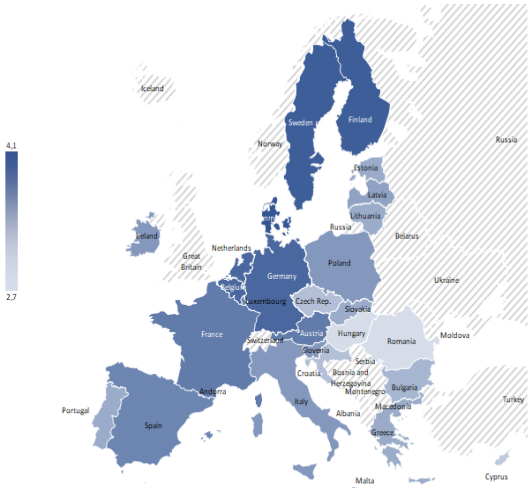
4. Analysis of the results

4.1. Spatial analysis of LPI

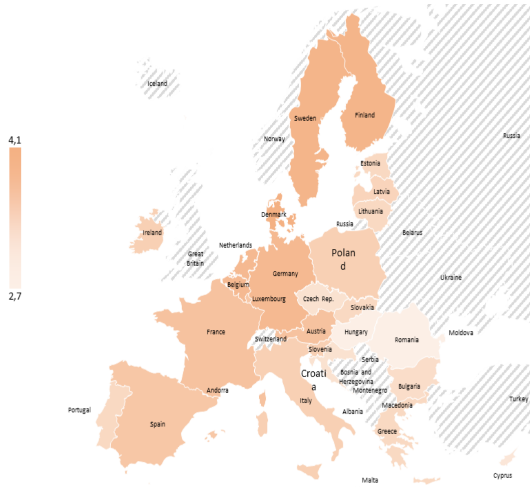
In this section various key performance indicators (KPIs) across different European Union countries in 2023 are analyzed. This analysis aims to highlight the disparities in performance and suggest potential strategies for enhancing efficiency and effectiveness in logistics operations.

Customs: When analyzing Fig. 2 (a), the first conclusion is that the countries with the best scores in 2023 for the Customs indicator are Sweden, Finland, and Denmark, with scores of 4 for Sweden and Finland, and 4.1 for Denmark. These high scores suggest that these countries have efficient customs processes, which likely contribute to smoother international trade and lower costs for businesses. On the other hand, the countries with the worst scores are Romania, Czech Republic, Cyprus, Croatia, and Hungary. These low scores indicate significant inefficiencies in customs procedures, which can lead to delays and increased costs.

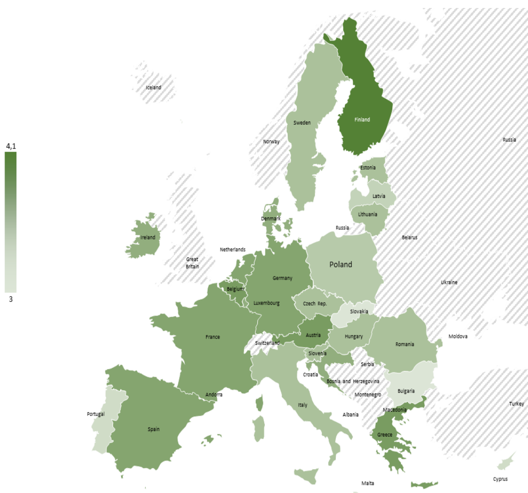
Infrastructure: the countries with the lowest quality of infrastructure are Romania, Cyprus, Croatia, and the Czech Republic (Fig. 2 (b)). Poor infrastructure quality in these countries can hinder economic growth and reduce the efficiency of logistics operations . Conversely, the countries with the best infrastructure quality are Sweden, Finland, Belgium, Germany, the Netherlands, and Denmark (Fig. 2 (b)). High-quality infrastructure in these countries supports efficient logistics and supply chain operations, contributing to their economic competitiveness.



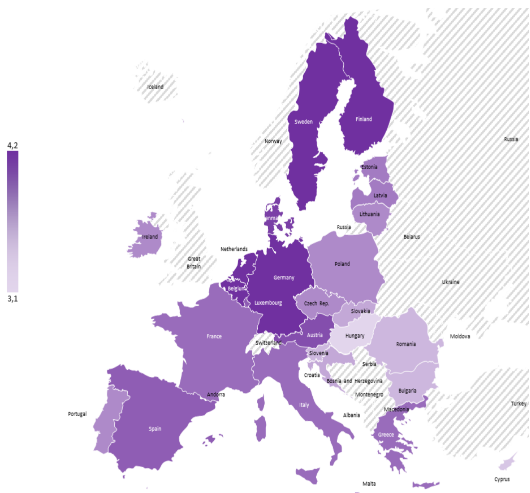
(a) Costums



(b) Infrastructure

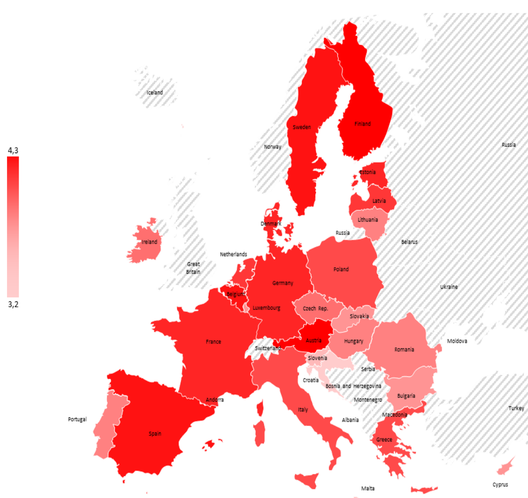


(c) International shipment



(d) Quality of Logistic Services

(e) Tracking and Tracing



(f) Timeless

Figure 2: *Indicators of LPI (2023)*

International Shipments: The countries where exporting at competitive prices is most difficult are Bulgaria, Slovakia, and Portugal. These scores suggest that businesses in these countries face challenges in maintaining competitive export prices, which can affect their global market presence. In contrast, the countries where competitive export prices are more achievable are Finland, Greece, Austria, and Belgium (see Fig. 2 (c)). These higher scores indicate a more favorable environment for exports, likely due to efficient logistics and supportive trade policies.

Quality of Logistics Services: Fig. 2 (d) shows the countries with the weakest scores in the Quality of Logistics Services indicator in 2023. The countries with the best scores are Denmark, Sweden, Germany, and the Netherlands. These high scores reflect the effectiveness and reliability of logistics services in these countries, which are crucial for maintaining supply chain efficiency and customer satisfaction. The countries with the weakest scores are Hungary and Cyprus. These low scores suggest that logistics services in these countries may be less reliable or efficient, potentially leading to higher costs and delays.

Tracking and Tracing: Fig. 2 (e) provides information on the countries with the highest and lowest scores for the Tracking and Tracing indicator. The countries with the highest scores are Austria, Germany, Finland, the Netherlands, Denmark, and Sweden. High scores in tracking and tracing indicate robust systems for monitoring shipments, which enhance transparency and reliability in the supply chain. The countries with the lowest scores are Slovenia, Lithuania, Czech Republic, and Portugal. These low scores suggest that these countries may face challenges in providing accurate and timely tracking information, affecting customer satisfaction and operational efficiency.

Timeliness: Analyzing Fig. 2 (f), the countries with the lowest scores for the Timeliness indicator in 2023 are Croatia, Malta, Slovenia, Bulgaria, Luxembourg, Cyprus, and Slovakia, Croatia, Malta, Slovenia, Bulgaria, Cyprus, Luxembourg, and Slovakia. These low scores indicate that shipments in these countries are often delayed, which can disrupt supply chains and increase costs. The countries with the highest scores are Belgium, Spain, Sweden, Austria, and Finland. High scores in timeliness reflect efficient logistics operations that ensure timely deliveries, which are critical for maintaining customer satisfaction and operational efficiency.

4.2. Weight criteria: CRITIC Mehod

In order to use the CRITIC method, it was necessary to clear the data, and selected the identifies the EU countries classified here as alternatives (rows of the decision matrix) ($A_i, i = 1, ...27$). Then, the 6 indicators (columns of the decision matrix) that compose the LPI were defined as criteria: customs (C_1), Infrastructure (C_2), International Shipment (C_3), Quality Services (C_4), Tracking and Tracing (C_5) and Timeliness (C_6). Matrix normalization is performed according to Equation (3). The values of the initial matrix and normalized values are displayed in Table 1.

Additionally the corresponding correlation matrix between criteria is calculated (Table 2).

The final results for the weights as step 4 defined in the previous section are in Table 3. It is possible to observe that criterium related to the international shipment(C_3) was the one has more relevance, while the quality services (C_4) was the lowest scored.

4.3. Ranking countries: MARCOS, SAW and TOPSIS Methods

After calculating the weights using the CRITIC method, the next step is to determine the ranking of each country. This was achieved by applying the formulas outlined in the previous section for the three selected methods: MARCOS, SAW, and TOPSIS.

Table 4 shows the final ranking for the three methods. The ranking of countries across the four methods: original ranking, MARCOS, SAW, and TOPSIS, offers valuable insights into their relative performances and the impact of different decision-making approaches. The rankings provide a comprehensive view of how countries compare in terms of their attributes,

	Initial decision matrix						Normalized decion matrix					
	C_1	C_2	C_3	C_4	C_5	C_6	C_1	C_2	C_3	C_4	C_5	C_6
AUT	3.7	3.9	3.8	4	4.2	4.3	0.714	0.733	0.727	0.818	0.923	1.000
BEL	3.9	4.1	3.8	4.2	4	4.2	0.857	0.867	0.727	1.000	0.769	0.909
BGR	3.1	3.1	3	3.3	3.3	3.5	0.286	0.200	0.000	0.182	0.231	0.273
HRV	3	3	3.6	3.4	3.4	3.2	0.214	0.133	0.545	0.273	0.308	0.000
CYP	2.9	2.8	3.1	3.2	3.4	3.5	0.143	0.000	0.091	0.091	0.308	0.273
CZE	3	3	3.4	3.6	3.2	3.7	0.214	0.133	0.364	0.455	0.154	0.455
DNK	4.1	4.1	3.6	4.1	4.3	4.1	1.000	0.867	0.545	0.909	1.000	0.818
EST	3.2	3.5	3.4	3.7	3.8	4.1	0.357	0.467	0.364	0.545	0.615	0.818
FIN	4	4.2	4.1	4.2	4.2	4.3	0.929	0.933	1.000	1.000	0.923	1.000
FRA	3.7	3.8	3.7	3.8	4	4.1	0.714	0.667	0.636	0.636	0.769	0.818
DEU	3.9	4.3	3.7	4.2	4.2	4.1	0.857	1.000	0.636	1.000	0.923	0.818
GRC	3.2	3.7	3.8	3.8	3.9	3.9	0.357	0.600	0.727	0.636	0.692	0.636
HUN	2.7	3.1	3.4	3.1	3.4	3.6	0.000	0.200	0.364	0.000	0.308	0.364
IRL	3.4	3.5	3.6	3.6	3.7	3.7	0.500	0.467	0.545	0.455	0.538	0.455
ITA	3.4	3.8	3.4	3.8	3.9	3.9	0.500	0.667	0.364	0.636	0.692	0.636
LVA	3.3	3.3	3.2	3.7	3.6	4	0.429	0.333	0.182	0.545	0.462	0.727
LTU	3.2	3.5	3.4	3.6	3.1	3.6	0.357	0.467	0.364	0.455	0.077	0.364
LUX	3.6	3.6	3.6	3.9	3.5	3.5	0.643	0.533	0.545	0.727	0.385	0.273
MLT	3.4	3.7	3	3.4	3.4	3.2	0.500	0.600	0.000	0.273	0.308	0.000
NLD	3.9	4.2	3.7	4.2	4.2	4	0.857	0.933	0.636	1.000	0.923	0.727
POL	3.4	3.5	3.3	3.6	3.8	3.9	0.500	0.467	0.273	0.455	0.615	0.636
PRT	3.2	3.6	3.1	3.6	3.2	3.6	0.357	0.533	0.091	0.455	0.154	0.364
ROM	2.7	2.9	3.4	3.3	3.5	3.6	0.000	0.067	0.364	0.182	0.385	0.364
SVK	3.2	3.3	3	3.4	3.3	3.5	0.357	0.333	0.000	0.273	0.231	0.273
SVN	3.4	3.6	3.4	3.3	3	3.3	0.500	0.533	0.364	0.182	0.000	0.091
ESP	3.6	3.8	3.7	3.9	4.1	4.2	0.643	0.667	0.636	0.727	0.846	0.909
SWE	4	4.2	3.4	4.2	4.1	4.2	0.929	0.933	0.364	1.000	0.846	0.909

Table 1: *Initial and normalized values for decion matrix for the CRITIC method, for the 27-EU countries in LPI context*

	C_1	C_2	C_3	C_4	C_5	C_6
C_1	1.000	0.931	0.557	0.896	0.757	0.662
C_2	0.931	1.000	0.580	0.884	0.754	0.672
C_3	0.557	0.580	1.000	0.685	0.681	0.619
C_4	0.896	0.884	0.685	1.000	0.827	0.806
C_5	0.757	0.754	0.681	0.827	1.000	0.861
C_6	0.662	0.672	0.619	0.806	0.861	1.000

Table 2: *Values for* r_{ij}

as determined by the weights calculated using the CRITIC method. This analysis highlights both consistencies and discrepancies among the rankings produced by the various methods.

The rankings demonstrate significant consistency in identifying the strongest and weakest performers across all methods. For example, Finland (FIN) consistently holds the top rank (1st) in all methods, indicating its dominant performance irrespective of the evaluation technique. This robustness suggests that Finland's attributes are well-aligned with the criteria and weights applied in the decision-making process.

Similarly, Bulgaria (BGR) and Cyprus (CYP) consistently occupy the bottom ranks (24th to 27th) across all methods. This unanimity highlights their relatively weaker performance on the evaluated criteria, making them less sensitive to the methodological differences.

While the top and bottom performers exhibit stability, countries in the middle tier show notable variability across methods. For instance, Denmark (DNK) ranks 2nd in the Original method but shifts to 3rd under MARCOS and SAW, and drops further to 6th in TOPSIS. This variability indicates that Denmark's performance is more influenced by the methodological nuances, particularly the treatment of ideal and negative-ideal solutions in TOPSIS.

Other middle-ranked countries, such as Ireland (IRL) and the Netherlands (NLD), also dis-

	C_1	C_2	C_3	C_4	C_5	C_6
σ_j	0.282	0.289	0.257	0.310	0.305	0.307
C_j	0.338	0.340	0.483	0.279	0.341	0.423
w_j	**0.153**	**0.154**	**0.219**	**0.127**	**0.155**	**0.192**

Table 3: *Values for standard deviation (σ_j). C_j and weights (w_j)*

	Score				Ranking			
Country	**Original**	**MARCOS**	**SAW**	**TOPSIS**	**Original**	**MARCOS**	**SAW**	**TOPSIS**
Austria	4.0	0.73	0.95	0.78	5	6	6	5
Belgium	4.0	0.74	0.95	0.82	5	4	4	2
Bulgaria	3.2	0.59	0.76	0.21	24	26	26	26
Croatia	3.3	0.60	0.78	0.30	19	22	22	21
Cyprus	3.2	0.58	0.75	0.17	24	27	27	27
Czech Republic	3.3	0.61	0.79	0.29	19	20	20	22
Denmark	4.1	0.74	0.95	0.78	2	3	3	6
Estonia	3.6	0.66	0.86	0.50	12	12	12	13
Finland	4.2	0.77	0.99	0.95	1	1	1	1
France	3.9	0.71	0.91	0.70	8	9	9	9
Germany	4.1	0.74	0.96	0.81	2	2	2	3
Greece	3.7	0.68	0.88	0.60	10	10	10	10
Hungary	3.2	0.60	0.77	0.26	24	25	25	24
Ireland	3.6	0.66	0.85	0.50	12	14	14	14
Italy	3.7	0.68	0.87	0.56	10	11	11	11
Latvia	3.5	0.64	0.83	0.42	16	16	16	16
Lithuania	3.4	0.63	0.81	0.36	17	17	17	17
Luxembourg	3.6	0.66	0.86	0.51	12	13	13	12
Malta	3.3	0.61	0.79	0.33	19	21	21	19
Netherlands	4.1	0.74	0.95	0.79	2	5	6	4
Poland	3.6	0.66	0.85	0.48	12	15	15	15
Portugal	3.4	0.62	0.80	0.33	17	18	18	20
Romania	3.2	0.60	0.77	0.26	24	24	24	23
Slovakia	3.3	0.60	0.78	0.25	19	23	23	25
Slovenia	3.3	0.61	0.79	0.35	19	19	19	18
Spain	3.9	0.71	0.92	0.71	8	8	8	8
Sweden	4.0	0.73	0.94	0.72	5	7	7	7

Table 4: *Scores and Rankings for LPI Methods*

play moderate shifts in rankings. For example, the Netherlands moves from 2nd in the Original method to 5th in MARCOS, 6th in SAW, and 4th in TOPSIS. This reflects the sensitivity of their rankings to how criteria weights and normalization are applied in each method.

Figure 3 highlighting the comparison of rankings obtained using various methods. From the methodologycal point of view, MARCOS and SAW methods generally align closely with the original rankings, with only minor deviations. This alignment can be attributed to their additive or weighted-sum approaches, which emphasize cumulative performance without heavily penalizing deviations from an ideal solution. Conversely, TOPSIS introduces more pronounced shifts, particularly among middle-tier countries, due to its dual focus on proximity to the ideal solution and distance from the negative-ideal solution.

The consistency among methods for certain countries, alongside variability for others, underscores the importance of selecting an appropriate decision-making method based on the specific context and objectives. For example, TOPSIS may be more suitable for scenarios requiring a balanced evaluation of both positive and negative aspects, while MARCOS or SAW may be preferred for simpler additive aggregation.

The rankings provide critical insights for stakeholders and policymakers, particularly in understanding the relative positions of countries and the potential impact of different evaluation methods. For countries consistently ranked at the top or bottom, the results validate their performance as either strong or weak. However, for countries with variable rankings, further

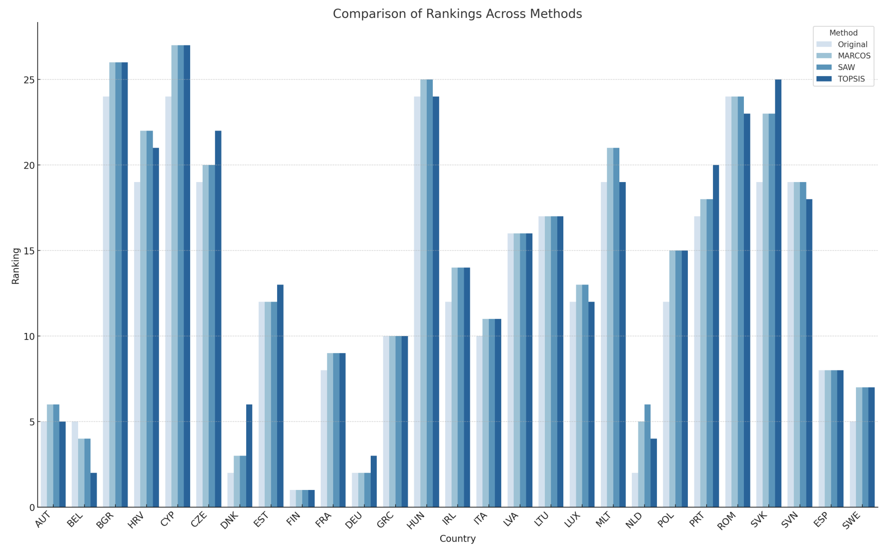


Figure 3: *Comparison of Country Rankings Across Different Decision-Making Methods*

analysis may be required to understand the underlying factors influencing their performance.

The LPI classified throught the two stage methodology CRITIC-MARCOS methods could be summarized in Figure 4. Finland and part of Western Europe are in the top of the ranking, while Central and Eastern Europe have the lower positions. Analyzing neighboring countries in relation to logistics indicators is crucial for several reasons. First, is important to think about trade efficiency, because when a company undestand the logistics infrastructure and capabilities of neighboring countries can help identify the most cost-effective routes for transporting goods, and could decrease significantly transit times. Second, understand what happens in different countries could increase the market access, specially, if regional trade agreements are prepared. Third issue is related to the resilience of supply chain, because by understanding the logistics capabilities of neighboring countries, businesses can diversify their supply routes, reducing dependency on a single route or country and enhancing resilience against disruptions. Finnaly, knowing what happen in different counties could lead to the increase of economic competitiveness, reducing the overall cost of goods sold, making products more competitive in the global market.

5. Conclusion

Analyzing the Logistics Performance Index within the European Union is vital for optimizing supply chains, enhancing market access, boosting economic competitiveness, improving supply chain resilience, informing policy and infrastructure development, supporting sustainable logistics, enhancing global trade competitiveness, and promoting economic growth, particularly for SMEs.

When the ranking for the three methods are provided, it is possible to see that Finland (FIN) consistently ranks first across all methods, demonstrating its robustness as the top performer. Similarly, Bulgaria (BGR) and Cyprus (CYP) are consistently ranked among the lowest, highlighting their weaker relative performance.

For middle-tier countries, such as Denmark (DNK) and the Netherlands (NLD), significant variability in rankings is observed, particularly in TOPSIS compared to the Original and additive methods (MARCOS and SAW). This suggests that the methodological differences, especially the treatment of proximity to ideal and negative-ideal solutions in TOPSIS, can notably influence rankings for countries with intermediate performance.

As future work, we intend to study the LPI, using other multi-criteria decision models and understand whether there are significant changes in the rankings presented.

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A hybrid approach to approximate the Pareto front of the MOST problem

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Abstract. This study introduces a hybrid NSGA-II algorithm with Multi-VNS for approximating the Pareto front of the multi-objective spanning tree (MOST) problem, building on recent approaches that have adapted NSGA-II combined with local search heuristics. By exploiting the property that a spanning tree is acyclic and that the addition of an edge generates a unique cycle, our mutation operator adds edges to a given spanning tree T of a connected graph G, thereby reducing the size of the MOST problem. By applying the exact mutation operator with low probability, this reduced problem is solved, producing a set of mutant solutions. The NSGA-II selection operator then approximates the Pareto front, which is further refined by a Multi-VNS metaheuristic to balance diversification and intensification. Comparative experiments with both exact and approximate methods demonstrate promising results.

Keywords: hybridization methods, metaheuristics, multi-objective spanning tree problem

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1. Introduction

Multi-objective optimization problems aim to identify efficient solutions that consider multiple, often conflicting objectives. One prominent example is the Multi-Objective minimum Spanning Tree (MOST), which is applied in various systems like communication networks, electric power systems, drainage systems, physical systems design, reducing data storage, cluster analysis. The authors in [17] applied a MOST approach under uncertainty conditions to optimize the distribution of petroleum products.

Several studies highlight the effectiveness of evolutionary and hybrid metaheuristics in approximating Pareto fronts for complex multi-objective problems. For example, [4] shows how advanced evolutionary algorithms can capture trade-offs among conflicting objectives using real-world data, underscoring the relevance of hybrid strategies for the MOST problem.

The MOST problem is well known to be NP-hard [5], making exact methods impractical for large instances. Consequently, approximate methods have been developed to produce a Pareto front that closely approximates the exact front for the MOST problem, providing feasible solutions within reasonable time frames.

We consider an undirected connected graph where each edge has an associated cost-vector of dimension $r \geq 2$. A cost-vector Z of dimension r dominates another cost-vector W of the same dimension, if Z is at least as good as W in all dimensions and they are not equal. In this case,

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a spanning tree T of G is Pareto optimal if its vector weight is not dominated by any vector weight of another spanning tree of G. The image of the Pareto optimal set forms the Pareto front set.

The main objective of this paper is to present a method for generating non-dominated points for the MOST problem that can effectively approximate the Pareto front set. The proposed algorithm, which combines the strengths of NSGA-II and "Variable Neighborhood Search" (VNS) [15], is referred to as the HMOST-Algorithm. The algorithm utilizes a two-point crossover operator and a $k-opt$ procedure as a mutation operator with a low probability. Subsequently, the VNS algorithm is used to refine a subset of points on the Pareto front generated by NSGA-II. Computational experiments demonstrate that the HMOST-Algorithm outperformes previous methods in terms of speed, scalability in solving instances with complete graphs of more than 200 nodes, and its ability to discover both supported and non-supported spanning trees.

The rest of the paper is organized as follows. Section 1 provides the basic concepts and notations used throughout this work. The procedures of the proposed HMOST-Algorithm are reported in Section 2. Section 3 reports numerical experimentation performed on instances described in [20] and other randomly generated instances. Finally, Section 4 concludes the study.

2. Background concepts

Consider a simple connected and undirected graph $G = (V, E)$ of order n, $V = \{v_1, v_2, \ldots, v_n\}$ is a set of vertices and $E = \{e_1, e_2, \ldots, e_m\}$ is a set of edges. Each edge $e_i \in E$, with $i \in \{1, \ldots, m\}$, is valued by a cost-vector $c(e_i) = (c_{ik})$, $k \in \{1, \ldots, r\}$, $r \geq 2$.

Example: Consider a connected undirected graph with 6 vertices and 13 edges, where each edge is weighted according to three criteria. The mathematical model associated with the

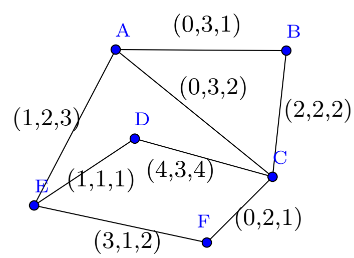

Figure 1: *A connected graph.*

MOST problem is expressed as follows:

$$(P)\begin{cases} Min\ C_k(x) = \displaystyle\sum_{i=1}^{m} c_{ik}x_i, & k \in \{1, \ldots, r\} \\ \displaystyle\sum_{i=1}^{m} x_i = n-1, & \\ \displaystyle\sum_{e_i \in E(S)} x_i \leq |S| - 1, & \forall S \subset V, S \neq \emptyset \\ x_i \in \{0,1\}, & \forall j \in \{1, \ldots, m\}, \end{cases} \quad (1)$$

where $x = (x_i)_{i \in \{1,\ldots,m\}}$, $x_i = 1$ if the edge $e_i \in T$, and $x_i = 0$ otherwise. $E(S)$ is the set of edges of the subgraph induced by S, $S \subset V$.

Let $T \subset E$ a spanning tree of G. The cost-vector of T is given by $C_k(T) = \sum_{e_i \in T} c_{ik}, \quad k \in \{1, \dots, r\}$. We denote $C(T) = (C_k(T))_{k \in \{1,\dots,r\}}$.

- The vector $C(T)$ dominates another vector $C(T')$ if $C_k(T) \leq C_k(T')$ for all $k \in \{1, \dots, r\}$, and $C_{k_0}(T) < C_{k_0}(T')$ for at least one index $k_0 \in \{1, ..., r\}$.
- T is efficient if there is no spanning tree T' of G such that $C(T')$ dominates $C(T)$.
- For any efficient spanning tree in the decision space, there is a corresponding non-dominated point in the criteria space. However, multiple efficient spanning trees can correspond to the same non-dominated point. The set of all efficient spanning trees is called the Pareto optimal set and the corresponding set of non-dominated points is called the Pareto front set. Let $\overline{T}$ the following set of edges: $\overline{T} = \{e \in E : e \notin T\}$. For each edge e belonging to $\overline{T}$, the set of edges $T \cup \{e\}$ contains a unique cycle μ_e and the set $B = \{\mu_e, e \in \overline{T}\}$ constitutes a cycle-basis of the graph G.

3. Related work

The multi-objective spanning tree problem has attracted much interest due to its practical applications in various fields. Researchers have developed numerous algorithms and heuristics to tackle this problem, ranging from exact methods like branch-and-bound and dynamic programming to approximate approaches such as genetic algorithms and multi-objective evolutionary algorithms. The challenge lies in balancing the trade-offs between different objectives to find a set of Pareto-optimal solutions that provide decision-makers with a range of trade-off choices. Furthermore, the complexity of the problem increases with the size of the graph and the number of objectives, making it a rich area for ongoing research and development.

In [22], an adaptation of the Genetic Algorithm (GA) is developed for the bi-objective minimum spanning tree problem using Prüfer number encoding [12], along with an early version of the Non-dominated Sorting Genetic Algorithm (NSGA) for giving out Pareto optimal spanning trees as closely as possible to the ideal point. To evaluate the performance of their adapted GA, the authors also proposed an algorithm intended to enumerate all Pareto-optimal spanning trees. However, [16] points out that this enumeration algorithm is incorrect and criticize it for failing to accurately compute the non-dominated solutions. They propose corrections and improvements, including a new enumeration method and an enhanced non-generational genetic algorithm for the MOST problem using a novel crossover operator (Dislocation Crossover) and a niche evolution strategy.

In [14] a novel genetic algorithm for the MOST problem is proposed while preserving the topoligical propreties of the graph. We note that algorithms [14, 16] have been tested on small graphs with two and three criteria. A particle swarm algorithm to solve problems with more than two objectives is presented in [13], but no experimental results are provided.
The authors in [8] introduce a Knowledge-based Evolutionary Algorithm (KEA), which demonstrates superior performance compared to the "Non-Sorting Genetic Algorithm" (NSGA-II) [9] algorithm in terms of both spread and the number of solutions identified. They initialize the population by leveraging extreme points to create an elite set of parents. Subsequently, the algorithm employs knowledge-based mutation exclusively to explore and uncover the remaining solutions on the Pareto front. A marking scheme is suggested to mitigate the reproduction of identical solutions and avoid dominated regions within the search space.

The study in [19] explores how local search techniques can enhance the overall effectiveness of NSGA-II for tackling the MOST problem, yet it lacks a comparative analysis against other existing methods in the literature. The authors evaluated the performance of three general-purpose local searches (Pareto Local Search, Tabu Search and Path Relinking) adapted to the multi-objective approach. Experimental results particularly with two and three objective functions, show that incorporating Pareto Local Search (PLS) in NSGA-II (NSGA-II+PLS) provides better performance.

To the best of our knowledge, these last two methods described above represent the most recent approximate approaches capable of handling more than two criteria and finding supported and non-supported Pareto solutions for large instances. This strongly motivated us to use them as references in our comparative study.

The survey presented in [11] reports the outcomes of a computational experiment featuring complete and grid graphs. It analyses the distinctive characteristics of each algorithm and evaluates the computational resources required to solve the instances.
Several exact methods have been proposed to solve the MOST problem. One such method, proposed in [3], generates all non-dominated points for problems involving at least two criteria. This branch-and-bound algorithm involves two main steps: (1) separating edges shared by at least two cycles, which allows the formulation of constraints as linear equalities, and (2) adding sub-cycle elimination inequalities to prevent cycles. Each branch of the search tree generates a multi-objective linear program with binary variables ($MOLBP$), representing the MOST problem at that branch.

The method in [6] targets only the supported solutions of the MOST problem. It begins by finding a lexicographical solution, then uses a neighborhood search starting from this optimal solution to identify the remaining supported spanning trees and their associated indifference regions. The study in [1] proposes a two-phase algorithm for the bi-objective minimum spanning tree problem. The first phase identifies extreme supported efficient solutions by combining mathematical programming with algorithmic techniques. The second phase identifies the remaining efficient solutions through a recursive edge interchange process, utilizing ascending non-zero reduced costs from weighted linear programs.

The method in [18] redefines the MOST problem as a One-to-One Multi-Objective Shortest Path (MOSP) problem, represented by a transition graph. The graph size is reduced using cost-based pruning criteria, and the Implicit Graph Multi-Objective Dijkstra algorithm is applied to solve the optimized MOSP instance, leveraging recent algorithmic advancements.

4. Description of the proposed method

One of the main reasons why population-based metaheuristics are favored over trajectory-based metaheuristics in multi-objective optimization is that they work with multiple solutions simultaneously, unlike traditional approaches that focus on a single solution. To address this disparity, we propose using the front obtained during the NSGA-II search process as the initial population for the VNS metaheuristic. The motivation for the hybrid algorithm is to combine the strengths of both methods to improve optimization performance.

Given the known convergence properties and limitations of NSGA-II, integrating the exact mutation operator, as well as the Multi-VNS framework, enhances the search dynamics. Specifically, this hybridization increases the algorithm's ability to improve the density of solutions along the obtained front and promotes more effective convergence toward the Pareto front. By combining these algorithms, the hybrid approach achieves a balance between exploration and exploitation, increases convergence speed, and improves the overall quality of solutions.

The proposed algorithm is outlined in the following procedures, describing the distinct steps of our algorithm, called the HMOST-Algorithm.

4.1. Encoding scheme

Initially, each edge in the graph $G = (V, E)$ consisting of n vertices and m edges, is assigned a unique identifier or number. The direct encoding of a chromosome is a set of $n-1$ dimension, wherein each element represents the associated number of an edge in the corresponding spanning tree. Prüfer coding is an encoding method using a sequence of $n-2$ natural numbers for a spanning tree as proposed in [22]. The authors in [16], point out that the direct encoding is better and more efficient than Purfer-based encoding.

4.2. Starting population

The initial population, comprising s spanning trees, is generated through:
1- The optimal spanning trees corresponding to each criterion,
2- Random generation with weighted-sum method,
3- Applying Kruskal's algorithm by randomly selecting edges of the graph G.

It is worth noting that the first two procedures exclusively produce supported solutions, contrasting with the third, which specifically seeks non-supported solutions. Furthermore, this approach consistently ensures the attainment of feasible solutions and a well-suited equilibrium between individual quality and diversity.

4.3. Crossover operator

The reproduction process involves combining two individuals through the crossover operator. This is achieved by randomly generating a number, denoted as ρ, within the interval [0,1]. Crossover occurs only if $\rho \leq 0.80$. In such instances, two-point crossover is employed:

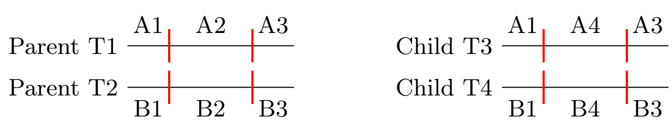

Figure 2: *2-point crossover.*

Two cut-points are randomly generated. The obtained offspring chromosomes do not correspond necessarily to spanning trees. To overcome this infeasibility, procedures of rearrangement of edges are introduced.

The two-point crossover is explained as follows, based on Figure 2:
Select edges of $A4$ in ascending order of the sums of costs in the set: $T2 \cup E \setminus (A1 \cup A3)$. Those of $B4$ are chosen in ascending order of the sums of costs in the set: $T1 \cup E \setminus (B1 \cup B3)$.

4.4. Diversified mutation operator

The mutation is achieved as in the biological evolutionary process; its priority aim is to generate better solutions in the sense of dominance. The diversity aspect is still present even in the case of non-improvement. The mutation operator is applied to the offsprings of the current population with a low predetermined probability $pm = 0.20$. The task involves creating a partial graph H from an offspring T by adding k edges from $E \setminus T$, $k \in \{1, 2, 3\}$. Hence, H contains k cycles. We then consider the subgraph L of H formed by only these cycles. The process of generating mutated offspring by adding k edges is called k-opt mutation.

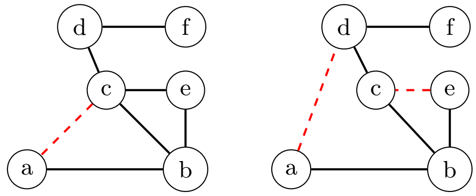

Figure 3: *Illustration of 1-opt (left): one edge added to form a single cycle L; and 2-opt (right): two edges added to form two cycles building the subgraph L. Red dashed edges are candidates added during mutation.*

Applying the 1-opt mutation, the subgraph L is reduced to an elementary cycle μ. If μ contains a dominated edge e, we obtain a spanning tree of L. This results in a spanning tree T' in H, $T' = E(H) \setminus \{e\}$, where $E(H)$ is the edge set of H. When we apply 2-opt mutation and 3-opt mutation, the subgraph L has a reduced dimension compared to the initial graph G. In these cases, we use our exact method, as described in [3], to generate all spanning trees in subgraph L corresponding to non-dominated points. As with 1-opt mutation, we reconstruct all non-dominated points for the partial graph H by sequentially adding the edges of $H \setminus L$ for each non-dominated points found previously. Unlike the 1-opt mutation, which is executed with high probability, the 2-opt mutation and 3-opt mutation are executed with low probability to minimize execution time.

Compared to standard heuristic mutations, this exact mutation operator ensures precision and can yield higher-quality offspring. However, due to computational cost, 2-opt and 3-opt are used less frequently.

Algorithm 1: k-opt Diversified Mutation

Spanning tree T, edge set E, parameter k the set of spanning trees T' $H \leftarrow T \cup$ random k edges from $E \setminus T$;
Identify subgraph L of H containing the k cycles;
$k = 1$ Find dominated edge e in the cycle;
$T' \leftarrow E(H) \setminus \{e\}$; Enumerate non-dominated trees in L using exact method;
non-dominated $T_L \subseteq L$ $T' \leftarrow T_L \cup (H \setminus L)$; the set of spanning trees T'

4.5. Dynamic selection operator

The selection operator starts by dividing the current population (with at least $2s$ individuals) into dominance fronts. The first front consists of non-dominated individuals, and subsequent fronts are formed by removing individuals from the previous front while maintaining non-dominance. The new population is created by selecting individuals according to the dominance order of the fronts. If additional individuals are needed to fill the population size s, individuals from front F_l ($l \geq 2$) are selected based on their "crowding" distance. If the first front is sufficient to fill the population, the size remains dynamic, determined by the first front's size.

4.6. Multi-VNS strategy

Using the set of spanning trees generated by the adapted NSGA-II algorithm, labeled MST, and the set ND representing its non-dominated points, a Multi-VNS-based algorithm generates neighboring solutions aimed at improving the set MST in order to obtain the set ND_{HMOST}, which contains non-dominated points for the HMOST-Algorithm. The ND_{HMOST} set is updated according to the Pareto dominance relation, and the populations obtained do not necessarily have the same size. The steps of the improvement by the Multi-VNS algorithm are described as follows:

- The set $P(MST)$ is a subset of the efficient solutions constructed by the NSGA-II algorithm, randomly chosen and having a cardinality of 10.
- The set N_k consists of neighborhood structures for $k = 1, 2, 3$. Given a spanning tree T (where $T \in P(MST)$), k edges are randomly selected from $E \setminus T$ and added to T. Consequently, k elementary cycles are created in the partial graph H, which is composed of the edges of T and the k added edges.

Each neighbor of T, according to N_k, can be reached by changing exactly k edges, with each edge belonging to a different elementary cycle of H.

Algorithm 2: Neighborhood Structure N_k

T: Spanning tree, E: Set of edges T': A neighbor of T
Randomly select k edges $e_1, e_2, ..., e_k$ from $E \setminus T$;
each selected edge e_i Find the elementary cycle $\mu(e_i)$ in $H = T \cup \{e_1, ..., e_k\}$;
Identify an edge $f \in T$ in the cycle $\mu(e_i)$;
Create $T' := T \setminus \{f\} \cup \{e_i\}$; T'

Algorithm 3: Multi-VNS Procedure

$P(MST)$: Set of spanning trees MST, ND_{HMOST}: Sets of efficient spanning trees and associated non-dominated points each solution T in $P(MST)$ $P(MST) := P(MST) \setminus \{T\}$;
a fixed number of iterations is not satisfied Randomly select a neighborhood structure N_k, with $k = 1, 2, 3$;
Generate a solution T' using N_k;
$C(T')$ dominates or equals $C(T)$ $T := T'$;
Update MST and ND with respect to non-dominance; MST and $ND_{HMOST} := ND$

Algorithm 4: HMOST-Algorithm

$G = (V, E)$: graph, C: cost-vector, s: population size, p: number of generations MST: efficient spanning trees, ND_{HMOST}: non-dominated points Encoding scheme setup;
Generate starting population P;
Initialize generation counter $iter := 0$, $Q := \emptyset$;
$iter \leq p$ Apply crossover-repair operator on P, store offsprings in Q;
Apply diversified mutation operator to elements in Q; $P := P \cup Q$;
Apply dynamic selection operator using crowding distance to get next population L;
$P := L$;
$iter := iter + 1$; Determine set $ST(F1)$ of trees in first front $F1$ of P;
$MST := ST(F1)$;
Extract $P(MST)$ from MST;
Apply Multi-VNS to obtain MST and ND_{HMOST};
MST and ND_{HMOST}

5. Computational experiments

The goal of this work is to show that the hybridization of the adapted NSGA-II metaheuristic with an exact method used locally, coupled with Multi-VNS Strategy, is able to give a better approximation of the Pareto front of the MOST problem than adapting a basic metaheuristic.

Parameter Settings: All experiments are conducted using *MATLAB R2018a* on a laptop with an *Intel Core i5* processor and *8 GB of RAM* and run on connected graphs. The HMOST-Algorithm uses the following parameters:

Population size:	$s = 100$
Number of generations:	$p = 50$
Crossover rate:	$\rho = 0.80$
Mutation rate:	$pm = 0.20$
Neighborhood structures in Multi-VNS:	$k = 1, 2, 3$
Stopping criterion (Multi-VNS):	20 iterations.

5.1. Comparison with the Pareto front

The first part of the experimental study is dedicated to the comparison between the Pareto front approximated by our proposed method and the Pareto front of the problem instance. Let ND_{HMOST} be the set of non-dominated points produced by the HMOST-Algorithm, and let $\alpha = \frac{|ND_{HMOST}|}{|ND|}$, where ND denotes the set of all non-dominated points. This ratio measures how well the HMOST-Algorithm approximates the Pareto front.

5.1.1. Instances description and computational results

1- Benchmark instances [20]: the results presented in Table 1 show that our approximate approach, HMOST-Algorithm, is able to recover on average 81.83% of the globally non-dominated solutions (i.e., those belonging to the Pareto front obtained by the exact method) in the three-criteria case and 80.16% in the four-criteria case. Approximately 20% of the remaining solutions are dominated.

Tests	n	m	$r = 3$			$r = 4$		
			$\|ND\|$	$\|ND_{HMOST}\|$	α	$\|ND\|$	$\|ND_{HMOST}\|$	α
$b01$	50	62	110	138	0.82	310	299	0.79
$b05$	50	100	712	946	0.80	923	1063	0.82
$b07$	75	94	340	377	0.83	402	410	0.80
$b10$	75	150	541	910	0.81	613	987	0.80
$b13$	100	125	101	120	0.84	304	412	0.81
$b14$	100	125	87	103	0.81	283	318	0.79

Table 1: *Comparison between Pareto front and HMOST-Algorithm for $r = 3$ and $r = 4$.*

2- SPACYC-Based Instance Generator [21]: The benchmark instances used in this study were kindly provided in the referenced article. We used 3 and 4 criteria with graphs ranging from 5 to 14 nodes. For each n, graphs with $m = n \cdot d$ edges, where $d \in \{5, 10, 15, 20\}$, were generated. The cost vectors were independently drawn from a uniform distribution over $[0, 100]$.

Table 2 shows that HMOST-Algorithm identified $81,7\%$ of non-dominated points for 3 criteria and $81,5\%$ for 4 criteria in average.

n	$r=3$			$r=4$		
	$\|ND\|$	$\|ND_{\text{HMOST}}\|$	α	$\|ND\|$	$\|ND_{\text{HMOST}}\|$	α
5	16.65	18.30	0.83	29.30	38.03	0.83
6	42.90	58.08	0.89	118.60	124.44	0.84
7	92.10	112.00	0.81	269.65	285.62	0.83
8	135.15	160.71	0.80	661.60	622.67	0.79
9	256.80	245.56	0.82	1370.95	1404.47	0.81
10	458.20	508.10	0.82	3629.15	3283.79	0.86
11	549.10	601.12	0.81	5426.50	6971.32	0.85
12	910.05	920.96	0.81	10985.80	12437.51	0.79
13	1206.20	1034.67	0.80	14881.60	15509.83	0.80
14	1909.00	2102.32	0.78	24679.75	22368.53	0.75

Table 2: *Comparison between exact Pareto front and HMOST-Algorithm for $r=3$ and $r=4$.*

3- Random instances: Graphs are generated using the Erdős–Rényi model [10], where a fixed number of vertices is defined and edges are added with a given probability p in which a fixed number of vertices is specified and edges are added with a given probability p. The cost vectors associated with the edges have dimensions 5 and 7, and their components are uniformly distributed within the interval $[-50, 100]$.

We can see that over 81% of non-dominated points have been found by the HMOST-Algorithm. We considered it useful to conduct an experiment on randomly generated instances in order to confirm the satisfactory results found previously, and indeed, the obtained results in Table 3 show that HMOST-Algorithm manages to generate nearly 80% of non-dominated points on average.

Tests	r	n	m	$\|ND\|$	$\|ND_{HMOST}\|$	α
Test1	5	20	[46,55]	74,80	307,20	0,81
Test2	5	30	[55,65]	930,30	980,50	0,79
Test3	5	50	[65,75]	1300,40	1612,06	0,77
Test4	5	100	[125,130]	627,39	1031,02	0,82
Test5	5	300	[325,340]	1267,40	1560,46	0,78
Test6	5	400	[425,440]	3427,94	6036,48	0,80
Test7	7	7	21	342,10	485,00	0,80
Test8	7	8	28	478,12	677,81	0,82

Table 3: *Comparison between Pareto front and HMOST-Algorithm using random instances with 5 and 7 criteria.*

5.2. Comparison with approximate methods

Comparative studies with randomly generated instances of cost-vectors of dimensions two, three and five, uniformly distributed in the interval $[-50, 100]$ are carried out for the three algorithms. The results are presented in Tables 5 and 6 respectively.
It should be noted that the number p of iterations is fixed as follows: 30 populations for the adapted NSGA-II procedure, 20 for the Multi-VNS strategy applied to each solution T in $P(MST)$, and 50 for each of the KEA and (NSGA II+PLS) algorithms.

Aspect	**HMOST**	**KEA**	**NSGA-II + PLS**
Chromosome encoding	edge-set	edge-set	Prüfer encoding
Extreme MSTs in the initial population	yes	yes	yes
Crossover operator	two-point crossover	not used	n-point crossover
Crossover probability	0.8	0	0.8
Mutation operator	k-opt, $k = 1, 2, 3$ solved by exact multi-objective method	knowledge-based mutation	controller-random mutation
Mutation probability	≤ 0.2	1	0.02
Hybridization strategy	NSGA-II + locally exact method + VNS	NSGA6II + k-best	NSGA-II + Pareto Local Search

Table 4: *Comparison of algorithmic innovations in HMOST, KEA, and NSGA-II+PLS.*

1 - Comparison using set coverage metric

This metric is used to compare two sets of potentially efficient solutions. The reference set, denoted REF, represents the set of non-dominated points obtained by combining the solutions from the two methods being compared.

The results are compared on average using the proportional measure to calculate the number of obtained non-dominated points of a given method belonging to the REF set [7].

We denote by $\alpha_i = (NDi \cap REF)/(REF)$ the proportion of solutions of the ND_i set belonging to REF set for $i \in \{HMOST, KEA, NSGAPLS\}$, where ND_{HMOST}, ND_{KEA} and $ND_{NSGAPLS}$ are the sets of non-dominated points obtained by HMOST, KEA-Algorithm and (NSGA-II+PLS)-Algorithm), respectively.

For each problem instance, we compare the non-dominated points produced by the two algorithms under evaluation.

K_n	r	$\|REF\|$	$\|ND_{HMOST}\|$	α_1	$\|ND_{KEA}\|$	α_2
$K80$	3	1700,02	1453,25	0,88	989,05	0,25
$K100$	3	4428,27	4324,80	0,73	3843,50	0,28
$K200$	3	9238,91	7277,00	0,87	7986,24	0,31
$K80$	5	3320,81	2878,10	0,86	2230,69	0,25
$K100$	5	8026,11	6695,08	0,83	4870,14	0,30
$K200$	5	10027,00	8783,60	0,87	6960,65	0,23

Table 5: *Comparison between HMOST-Algorithm and KEA-Algorithm.*

K_n	r	$\|REF\|$	$\|ND_{HMOST}\|$	α_1	$\|ND_{NSGAPLS}\|$	α_3
$K80$	2	1528,43	1213,9	0,85	1001,31	0,32
$K100$	2	5034,13	4421,76	0,82	3843,50	0,29
$K80$	3	2616,77	2027,08	0,79	1109,65	0,38
$K100$	3	9123,45	7392,92	0,84	5769,23	0,31

Table 6: *Comparison between HMOST-Algorithm and (NSGA-II + PLS)-Algorithm.*

Each algorithm is executed ten times for all instances of the problem. Therefore, we have found it useful to present in each row of Tables 5 and 6 the average results obtained across all runs.

Notice that for all instances, the HMOST-Algorithm outperforms the KEA-Algorithm by grabbing 84% and the (NSGA-II+PLS)-Algorithm by grabbing 83% from the set REF. Hence, it determines a greater number of solutions in the REF set compared to the KEA and (NSGA-II+PLS) Algorithms whose contributions are only 27% and 32% respectively.

2 - Comparison using two performance metrics

To evaluate the quality of the approximated points, we rely on two widely used indicators: the Hypervolume (HV) and the Inverted Generational Distance (IGD).

Hypervolume (HV) [23]: measures the volume of the objective space dominated by the approximated front with respect to a reference point Z_{ref} (commonly $Z_{\text{ref}} = \{0\}^k$ for k objectives). It reflects both convergence to the Pareto front and solution diversity. A higher HV indicates better performance.

Inverted Generational Distance (IGD) [7]: quantifies the average distance from each point in the Pareto front ND to its nearest solution in the approximated set A. Defined as:

$$\text{IGD}(ND, A) = \left(\frac{1}{|ND|} \sum_{s \in ND} d_s^2 \right)^{1/2}. \tag{2}$$

where d_s is the Euclidean distance between $s \in ND$ and its closest point in A. Lower IGD values indicate better convergence.

a - Comparison with the Pareto front

Instance	Hypervolume (HV)			IGD		
	KEA	NSGA-II+PLS	HMOST	KEA	NSGA-II+PLS	HMOST
Test4	45037	43572	67001	0.42	0.55	0.23
Test5	108786	127659	138716	0.87	0.76	0.38
Test6	442700	413215	603604	1.25	1.32	0.60

Table 7: *Comparison of algorithms on Hypervolume (HV) and IGD indicators.*

In order to assess the convergence of the obtained Pareto fronts of the three methods to be compared towards the Pareto front, we computed the HV and IGD metrics values. The obtained results are resumed in Table 7. It is obvious that the obtained values HV and IGD of each instance using HMOST-Algorithm are better than those obtained by the other two algorithms.

b - Comparison with reference set

Instance	r	Hypervolume (HV)			IGD		
		KEA	NSGA-II+PLS	HMOST	KEA	NSGA-II+PLS	HMOST
K50	3	9.71	9.66	9.87	0.74	1.07	0.39
K100	3	39.02	40.75	41.78	2.73	2.18	0.69
K200	3	82.97	83.32	88.99	9.50	10.54	3.53
K50	4	5869.6	5596.8	7326.2	2.30	6.41	1.82
K100	4	86807.1	82412.5	93812.2	7.30	9.84	3.79
K200	4	85955.4	82399.7	98077.1	4.75	6.12	2.77

Table 8: *Comparison of algorithms on HV and IGD indicators (values in 10^5 for HV).*

Table 8, for both 3-objective and 4-objective instances, HMOST-Algorithmachieves the best performance in the majority of the tested cases. These results demonstrate the strong ability of HMOST-Algorithm to generate high-quality Pareto fronts with both better convergence and distribution, especially as the problem size and number of objectives increase.

6. Conclusion

This study aimes to provide a comprehensive depiction of the Pareto front for the MOST problem. To address this, we introduced the HMOST-Algorithm, which operates in two distinct phases. Initially, the hybrid approach combines the NSGA-II algorithm that uses locally an exact mutation operator to obtain the first Pareto front. In the second phase, we apply the Multi-VNS Strategy to improve a subset of the previous obtained solutions . By leveraging a specialized mutation operator together with the the Multi-VNS Strategy, the HMOST-Algorithm becomes a flexible optimization framework capable to controlling both phases through predefined probabilities.

A comparative study demonstrated that, on average, the proposed HMOST-Algorithm identifies nearly 81% of the non-dominated points. This confirms that the generated front closely approximates the Pareto front. In addition to overcoming the inability of the exact method to solve large instances, given the explosive complexity of the MOST problem, the proposed algorithm also demonstrates excellent performance compared to the KEA and (NSGA-II+PLS) algorithms in comparative studies on complete graphs with up to 200 nodes and cost-vectors with dimensionalities between 2 and 7.

Considering the experimental results, the HMOST-Algorithm consistently achieves lower IGD values, indicating stronger convergence toward the Pareto front. Additionally, it obtains higher HV values, reflecting better diversity and distribution of solutions across the objective space. This indicates that the Pareto front approximated by the HMOST-Algorithm is closer to the Pareto front and exhibits broader coverage compared to those obtained by the two cited algorithms, which, to the best of our knowledge, are the most recent available methods. However, while the algorithm performs well on graphs of up to 200 nodes, the computational cost of the exact k-opt mutation and Multi-VNS steps could become a bottleneck for very large-scale graphs (e.g., 1000+ nodes). Moreover, although KEA and NSGA-II+PLS are relevant benchmarks, additional recent multi-objective metaheuristics (e.g., NTGA2, theta-DEA, MOEAD) were not included in this comparative analysis. To further enhance the HMOST-Algorithm's scalability, it is advisable to explore parallelization of the exact mutation and Multi-VNS procedures. Additionally, investigating approximate k-opt variants or incorporating surrogate modeling techniques can help reduce the computational burden associated with large neighborhood structures.

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Reliability and performance evaluation of retrial repairable systems with an unreliable repairer using the $GSPN$ model

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Abstract. Generalized Stochastic Petri Nets ($GSPNs$) are widely used in reliability analysis to model and optimize the dependability of complex systems, enabling assessments of availability, failure probabilities, and repair processes through stochastic timing. This study investigates a repairable redundant retrial system incorporating warm and cold standbys and an unreliable repairer through a Generalized Stochastic Petri Nets. The repairer experiences both active (during repair) and passive (idle) breakdowns. Component and repairer lifetimes are exponentially distributed. A component that fails undergoes immediate repair if the repairer is available; otherwise, it enters a retrial $FCFS$ orbit. A Continuous Time Markov Chain ($CTMC$) derived from the $GSPN$ model yields steady state probabilities and availability. System reliability and time to first failure are then determined via Laplace transforms. Numerical examples demonstrate the system reliability performance.

Keywords: K-out-of-n : G system, reliability, generalized stochastic Petri net, standby redundancy

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1. Introduction

Reliability engineering focuses on designing and developing systems for consistent performance throughout their operational life with minimal probability of failure. It encompasses various methodologies, including failure analysis, risk assessment, and redundancy techniques, to ensure systems meet their reliability targets. A K-out-of-n : G system is deemed successful if a minimum of K units remain operational among the n components. These systems are frequently employed in fault-tolerant designs, where the objective is to maintain system functionality despite component failures. By incorporating redundancy, K-out-of-n systems help achieve this goal.

Standby redundancy enhances system reliability by having backup components that can seamlessly replace failed primary components. This redundancy is categorized as hot (backups continuously active), warm (pre-configured backups with minimal activation delay), or cold (inactive backups requiring manual initiation upon primary component failure). Hot standby components share the same failure rate with primary components, standby mode, failure rate of cold components is zero, while warm components failure rate is between the two. Readers interested in the topic are encouraged to refer to Barlow and Proshan [3], Birolini [4], David [20], Lai and Xie [17] and Kececioglu [15].

Our understanding of standby redundancy and K-out-of-n systems has been significantly advanced through numerous studies exploring reliability analysis, system configurations, and

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maintenance strategies. For example, Srinivas et al. [21] studied the system reliability under the (N, T) policy. The reliability of generalized repairable systems incorporating cold and warm components in standby are explored by Wang and Loman [24] and Ke and Wang [13] respectively. Yuan [25] examined availability and reliability in systems with multiple vacations. Other studies, such as, Amari [1] and Levitin and Amari [18], focused on reliability algorithms and failure time distributions for systems incorporating shared units in standby. Systems with retrial phenomena have been explored by Ke et al. [14] and Kuo et al. [16]. Bounds for reliabilities under probabilistic constraints are derived by Unuvar et al. [22]. In [12] a warm standby system is studied using the semi Markov process and regenerative point technique.

Generalized Stochastic Petri Nets ($GSPNs$) are bipartite graphs having the power to model complicated systems. They also describe and analyze stochastic concurrent systems with synchronization properties. They extend the capabilities of the classical Petri nets by using probabilities to represent the timing of events in complex systems. Graphically, a $GSPN$ consists of places (circles), tokens (black dots or numbers), and transitions. Transitions are of two types: immediate transitions (thin bars), which fire instantaneously, and timed transitions (rectangles), which have an exponentially distributed duration. Arcs connect transitions and places, indicating the circulation of tokens. Transitions are prevented from firing by inhibitor arcs. Timed transitions lead to tangible markings, while immediate transitions result in vanishing markings. A reachability graph shows all possible states (markings) of a $GSPN$, with nodes representing markings and edges representing transition firings. A $CTMC$ is isomorphic to $GSPN$ tangible reachability graph. A $GSPN$ has a steady-state probability distribution if its initial marking represents a home state (the system can return to it from any other marking) and it is bounded (has finite tokens in each place). $GSPNs$ are successfully used to model manufacturing systems [2, 23], computer science [6, 8, 28], business processes [5], queuing systems [7, 9, 10], and wireless networks [26, 27]. Besides these applications, $GSPNs$ are also widely used in reliability analysis of systems, providing powerful tools for modeling and evaluating system performance and dependability [30, 27, 11, 29]

The aim of our work is to examine a retrial K-out-of-n:G system incorporating warm and cold components in standby with an unreliable repairer using a Generalized Stochastic Petri Net.

The following is the paper's structure: Section 2 describes the K-out-of-n system incorporating warm and cold standbys and an unreliable repairer and presents the proposed $GSPN$ model. Section 3 delves into both steady-state and transient analysis, deriving key reliability indices system availability, reliability function, and the mean time to first failure ($MTTF$). Section 4 concludes with comprehensive illustrative examples that examine the influence of system variables on the reliability indices.

2. Model Description

We investigate a retrial K-out-of-$n : G$ system, composed of M uniform primary operational components, supported by W warm and C cold components in standby. The system operates as long as at least K components (out of the total $M + W + C$) are in operational mode (either actively operating or available in standby). The system units are prone to failure, and any failed component is repairable. Upon failure of a primary operating component, immediate replacement occurs using an available warm standby component, if one exists; otherwise, a cold standby component is activated as the new primary component. Similarly, if a warm standby component fails while it is on standby, it will be replaced immediately by an available cold component. Repairer failures can occur during active repair periods (active breakdowns) and during periods of inactivity (passive breakdowns). Upon failure, a component joins an $FCFS$ orbit for a random duration if the repairer is either undergoing repair or busy.

We assume that:

- Failures of primary and warm units are characterized by Poisson processes, with rates η and α ($\eta > \alpha > 0$) respectively,
- Upon failure, a unit is immediately repaired if the repairer is available.An exponential distribution with mean $\frac{1}{\mu}$ governs the repair times of all components,
- The repairer is subject to Poisson failures with rate δ when busy and θ when idle. The repairer recovery time is characterized by an exponential distribution with parameter σ,
- If the repairer is unavailable (busy or under repair), failed components join an $FCFS$ orbit. Subsequently, the repairer selects the next component from the orbit for service, with the selection time following an exponential distribution with mean $\frac{1}{\gamma}$,
- All stochastic processes are assumed to be independent.

2.1. Practical example

Underground mining operations generate significant amounts of toxic and dangerous gases, which not only have harmful effects on the personal safety and health of employees, but also force severe halts to mining activities. The mine must have complete and autonomous ventilation equipment, or mine ventilators, to provide adequate fresh air underground and to diffuse and remove dangerous and poisonous gases. Assume that this ventilation system has M working mine ventilators, W warm standby ventilators, and C cold standby ventilators. At least K ventilators must be operational to ensure the purity of underground air. If $M+W+C-K+1$ mine ventilators fail, the ventilation system fails. We assume that all mine ventilators are new at time $t = 0$. Operating and warm mine ventilators fail independently of each other. An available warm ventilator (or an available cold one) replaces a failed operating mine ventilator (or a failed warm one). When a failed mine ventilator finds the repairer available, it is repaired immediately and restored to perfect condition. Mine ventilators may experience breakdowns during either the repair service period or idle periods due to malfunctions or lifetime limitations and need to be repaired. Upon failure, a component is placed in a buffer (referred to as an orbit) and waits there if the repairer is busy or out of order. Once available, the repairer selects one of the failed mine ventilators from the orbit queue, if any are present. Consequently, a ventilation safety system with an unreliable repairer can be modeled as a retrial repairable system incorporating warm and cold standby components and an unreliable repairer.

2.2. *GSPN* description

Figure 1 depicts the $GSPN$ model of our retrial K-out-of-n system, incorporating warm and cold standbys and an unreliable repairer. The Places of the $GSPN$ are defined as the following:

- Primary operating components, Warm standby components and cold standby components are represented by places P_M, P_W and P_C with an initial marking of M, W and C receptively,
- Place S indicates whether a failed component, either newly failed or selected from the orbit, is ready for repair,
- The orbit is represented by place *Orbit*,
- Components under repair are represented by place *Rep*,

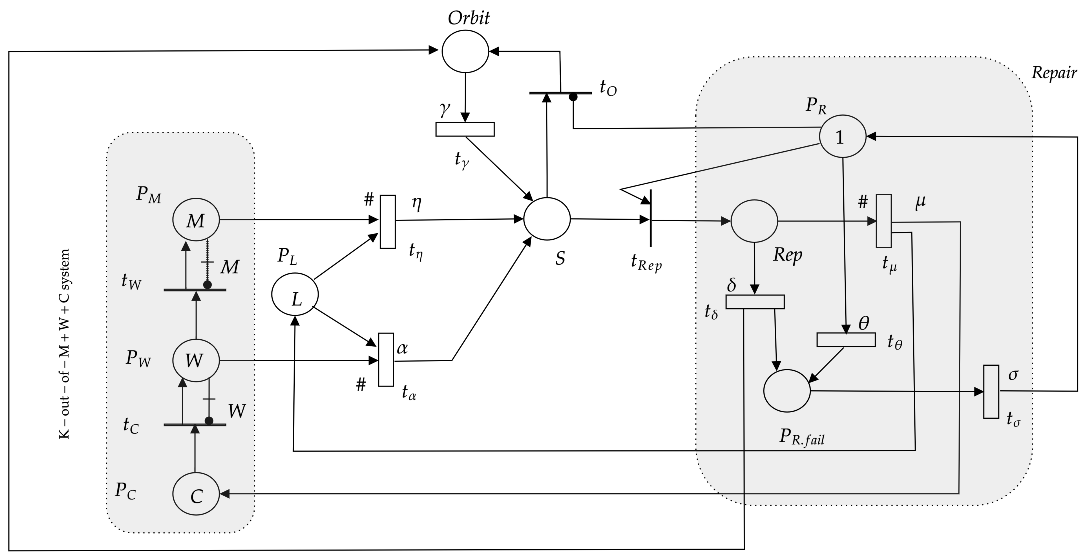


Figure 1: *GSPN model for the retrial K-out-of-n : G system incorporating warm and cold standbys and a single unreliable repairer*

- Failed repairer is represented by place $P_{R.fail}$. This place signifies that the repairer is unavailable due to a failure,
- Free repairer is represented by place P_R. This place indicates that the repairer is operational and ready to address component failures,
- Number of failed components: represented by place $P_L(L = M + W + C - K + 1)$. This place tracks the current number of components requiring repair.

Thus, the $GSPN$ starts with an initial marking M_0, which is:

$$M_0(\#P_M, \#P_W, \#P_C, \#P_L, \#Orbit, \#P_R, \#S, \#Rep, \#P_{fail}) = M_0(M, W, C, L, 0, 1, 0, 0, 0),$$

where # refers to the marking of a place p.

The failure of a primary or a warm component is modeled by the firing of transitions t_η and t_α with rates $\#P_M\eta$ and $\#P_W\alpha$ respectively. The arrival of a selected component from the orbit is characterized by the firing of transition t_γ with rate γ. The end of the repair period is represented by the firing of transition t_μ with rate μ. Active and passive breakdowns of the repairer are represented by the firing of transitions t_δ and t_θ with rates δ and θ respectively. The recovery period for the failed repairer is characterized by the firing of transition t_σ with rate σ. Transition t_{Rep} fires when place S contains one token (indicating a component ready for repair) and the repairer is available (P_R has a token). Transition t_O (indicates that the failed component is placed in orbit) fires if the repairer is under repair or busy. Transition t_W fires if P_M has fewer tokens than M (replacing a failed primary component with a warm component). Transition T_C fires if P_W has fewer tokens than W (replacing a failed warm component with a cold component). The place P_L tracks failed components, and when L components fail (resulting in P_L having 0 tokens), transitions t_η and t_α are disabled, indicating the retrial K-out-of-$n : G$ system incorporating warm and cold standbys and an unreliable repairer failure.

The set of places, transitions, and inhibitor arcs that define the structure and behavior of the Petri net model are formally presented in Tables 1, 2, and 3.

Place	Description
P_M	Represents primary operating components, initialized with M tokens.
P_W	Represents warm standby components, initialized with W tokens.
P_C	Represents cold standby components, initialized with C tokens.
S	Indicates whether a failed component (newly failed or selected from the orbit) is ready for repair.
Orbit	Represents the orbit where failed components wait to be selected for repair.
Rep	Represents components currently under repair.
$P_{R.fail}$	Signifies the failed state of the repairer (i.e., the repairer is unavailable due to failure).
P_R	Indicates a free and operational repairer ready to address component failures.
P_L	Tracks the number of failed components; $L = M + W + C - K + 1$.

Table 1: *Description of Places in the GSPN Model*

From Place	To Transition (Immediate)	Role of the Inhibitor Arc
P_M	t_W	Prevents the firing of t_W when $\#P_M = M$
P_W	t_C	Prevents the firing of t_C when $\#P_W = W$
P_R	t_O	Prevents the firing of t_O when $\#P_R = 1$

Table 2: *Inhibitor arcs and their roles (transition-focused)*

Transition	Type	Description	Firing time/rate
t_η	Timed	Failure of a primary component	Rate: $\#P_M\eta$
t_α	Timed	Failure of a warm standby component	Rate: $\#P_W\alpha$
t_γ	Timed	Selected component arrival from orbit	Rate: γ
t_μ	Timed	Completion of repair process	Rate: μ
t_δ	Timed	Active breakdown of the repairer	Rate: δ
t_θ	Timed	Passive breakdown of the repairer	Rate: θ
t_σ	Timed	Recovery of failed repairer	Rate: σ
t_{Rep}	Immediate	Repair begins if the component is ready and the repairer is free	Fires if $\#S = 1$ and $\#P_R = 1$
t_O	Immediate	Failed component routed to orbit	Fires if $\#P_{R.fail} = 1$ or $\#P_R = 0$
t_W	Immediate	Warm component replaces failed primary component	Fires if $\#P_M < M$
t_C	Immediate	Cold component replaces failed warm component	Fires if $\#P_W < W$

Table 3: *GSPN Transition Descriptions and Semantics*

3. Stochastic analysis

The goal of this section is to evaluate the steady-state and transient behavior of the retrial K-out-of-$M + W + C$ system incorporating warm and cold standbys and a single unreliable repairer and derive the main reliability indices including steady-state availability, reliability function and $MTTF$.

At time t, the marking of the system is characterized by three token counts: $F(t)$ in place *Rep*, $O(t)$ in *Orbit*, and $B(t)$ in $P_{R.fail}$. The $GSPN$ behavior is captured by the Markovian marking process $\{X(t), t \geq 0\} = \{(F(t), O(t), B(t)), t \geq 0\}$, defined on the state space $E = \{(i, j, n) : 0 \leq i \leq 1, 0 \leq j \leq L - 1, 0 \leq n \leq 1\} \cup (0, L, 0), (0, L, 1)$, with state-transition rate diagram depicted in Figure 2.

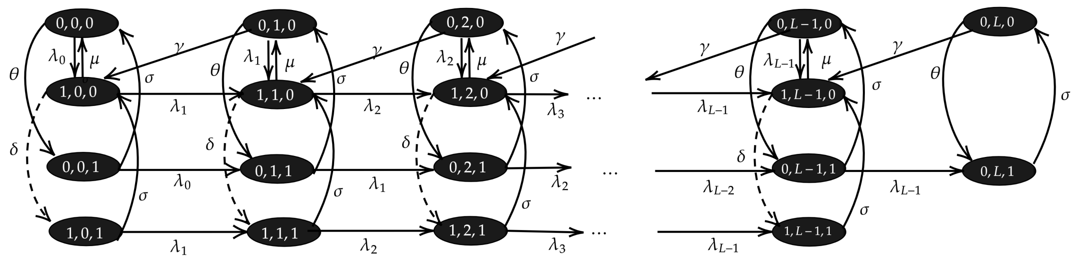
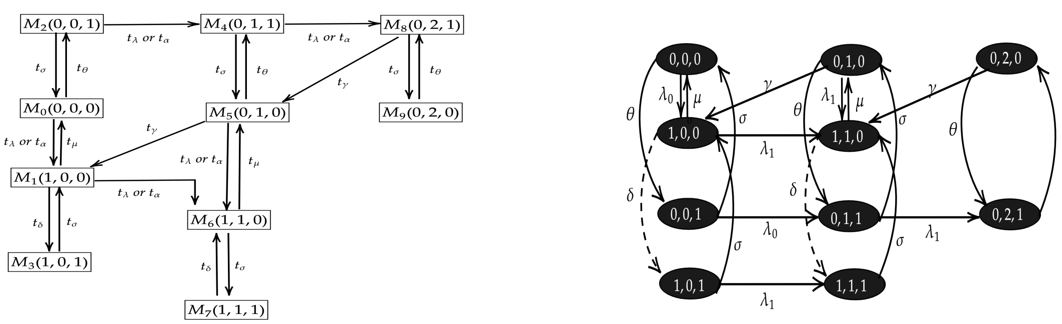


Figure 2: *State-transition diagram of the K-out-of-n;G retrial system incorporating warm and cold standbys and an unreliable repairer*

To clarify the structure of the state space and the system's dynamic behavior, we present a small-scale example with parameters $M = 2$, $W = 1$, $C = 1$, and $K = 3$, along with its corresponding reachability graph and associated $CTMC$, shown in Figures 3a and 3b, respectively



(a) *Reachability graph for the GSPN model with $M = 2$, $W = 1$, $C = 1$, and $K = 3$*

(b) *CTMC representation derived from the reachability graph in Figure 3a*

Figure 3: *(a) Reachability graph and (b) its corresponding CTMC for the GSPN model with $M = 2$, $W = 1$, $C = 1$, and $K = 3$.*

The transition rate matrix Q of the marking process, of order $4L+2 \times 4L+2$, is a tridiagonal block matrix given as follows:

$$Q = \begin{pmatrix} B_0 & C_0 & & & & \\ A & B_1 & C_1 & & & \\ & A & B_2 & C_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & A & B_{L-1} & C_{L-1} \\ & & & & A_L & B_L \end{pmatrix}_{4L+2\times 4L+2},$$

where

$$B_0 = \begin{bmatrix} -(\eta_0+\theta) & \eta_0 & \theta & 0 \\ \mu & -(\eta_1+\mu+\delta) & 0 & \delta \\ \sigma & 0 & -(\eta_1+\sigma) & 0 \\ 0 & \sigma & 0 & -(\eta_1+\sigma) \end{bmatrix},$$

$$B_j = \begin{bmatrix} -(\eta_j+\theta+\gamma) & -\eta_j & \theta & 0 \\ \mu & -(\mu+\delta) & 0 & \delta \\ \sigma & 0 & -\sigma & 0 \\ 0 & \sigma & 0 & -\sigma \end{bmatrix}, \quad j=0,1,\dots,L-1, \quad B_L = \begin{bmatrix} -(\theta+\gamma) & \theta \\ \sigma & -\sigma \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_L = \begin{bmatrix} 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C_j = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \eta_{j+1} & 0 & 0 & 0 \\ 0 & \eta_j & 0 & 0 \\ 0 & 0 & \eta_{j+1} & 0 \end{bmatrix}, \quad j=0,1,\dots,L-2, \text{and} \quad C_{L-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \eta_{L-1} \\ 0 & 0 \end{bmatrix}.$$

Given j failed units, the system instantaneous failure intensity η_j is as follows:

$$\eta_j = \begin{cases} M\eta + W\alpha, & j \in [0, C-1], \\ M\eta + (M+W-j)\alpha, & j \in [C, W+C-1], \\ (M+W+C-j)\eta, & j \in [W+C, L-1]. \end{cases}$$

3.1. Steady-State Distribution

The $GSPN$ model depicted in Figure 1 admits a steady-state probability distribution because its initial marking represents a home state and it is bounded. Let $P = (P_0, P_1, \dots, P_{L-1}, P_L)$ be the stationary probabilities vector where $P_j = (P_{0,j,0}, P_{1,j,0}, P_{0,j,1}, P_{1,j,1})$ $(j = 0, 1, \dots, L-1)$ and $P_L = (P_{0,L,0}, P_{0,L,1})$. The vector P satisfies the following matrix equations:

$$P_0 B_0 + P_1 A = \mathbb{O}_4, \tag{1}$$

$$P_{j-1} C_{j-1} + P_j B_j + P_{j+1} A = \mathbb{O}_4, \; j = 1, 2, \dots, L-1, \tag{2}$$

$$P_{L-1} C_{L-1} + P_L B_L = \mathbb{O}_2, \tag{3}$$

where $\mathbb{O}_4 = (0,0,0,0)$ and $\mathbb{O}_2 = (0,0)$.

We summarize the steady-state distribution $P_j (j = 1, 2, \dots, L)$, for the model in the following theorem:

Theorem 1. *For a retrial K-out-of-$M+W+C$: G system incorporating warm and cold standbys and an unreliable repairer with state space E, the steady-state probabilities $P_j, (j = 1, 2, \dots, L)$ are as follows*

$$P_0 = P_{0,0,0} r_0, \tag{4}$$

$$P_{0,0,0} = \frac{1}{r_0((\mathbb{I}_4 + \sum_{j=1}^{L} \prod_{m=0}^{j-1} R_i) e_4 + R_L e_2))} \quad , \tag{5}$$

$$P_j = P_0 \prod_{i=0}^{j-1} R_i, \; j = 1, 2, \dots, L, \tag{6}$$

and

$$P_L = P_{L-1}R_L \tag{7}$$

where $R_n = -C_n(B_{n+1} + R_{n+1}A)^{-1}$, $R_L = -C_{L-1}B_L^{-1}$, $\mathbb{I}_4$ *is the identity matrix,*

$e_2 = \begin{bmatrix}1\\1\end{bmatrix}$, $e_4 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ *and* $r_0 = (1, \frac{\eta_0}{\mu} + \frac{\theta\eta_1}{\mu(\eta_1+\sigma)}, \frac{\theta}{(\eta_1+\sigma)}, \frac{\delta}{(\eta_1+\sigma)}(\frac{\eta_0}{\mu} + \frac{\theta\eta_1}{\mu(\eta_1+\sigma)}))$.

Proof. Using equations (1)-(3) together with $\sum_{j=0}^{L-1} P_n e_4 + P_L e_2 = 1$, the result is derived through straightforward calculations and substitutions. □

Let A_∞ denote the steady-state availability, representing the steady-state probability of system operability. Hence, we have

$$A_\infty = 1 - (P_{1,L-1,0} + P_{1,L-1,1} + P_{0,L,0} + P_{0,L,1}) = 1 - P_{L-1}\begin{bmatrix}0\\1\\0\\1\end{bmatrix} - P_L\begin{bmatrix}1\\1\end{bmatrix}. \tag{8}$$

3.2. Reliability function

In this section, under the condition that at time $t = 0$ all system components are operational and new, our objective is to determine the system reliability $R(t)$ and the time to first failure $MTTF$. In order to achieve this, we modify the model by making the failure states $(0, L, 1)$, $(0, L, 0)$, $(1, L-1, 0)$ and $(1, L-1, 1)$ absorbing states (i.e., removing all transitions from these states to functioning states). This results in a new $CTMC$ $\{\tilde{X}(t), t \geq 0\}$, with state space $\tilde{E} = \{(i, j, n), 0 \leq i \leq 1, , 0 \leq j \leq L-2, 0 \leq n \leq 1\} \cup \{(0, L, 1)\}$. Its generator matrix $\tilde{Q}$ of order $4L + 1 \times 4L + 1$ is defined by:

$$\tilde{Q} = \begin{pmatrix} B_0 & C_0 & & & & \\ A & B_1 & C_1 & & & \\ & A & B_2 & C_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & A & \tilde{B}_{L-1} & \tilde{C}_L \\ & & & & & 0 \end{pmatrix}_{4L+1\times 4L+1},$$

where

$$\tilde{B}_{L-1} = \begin{bmatrix} -(\eta_{L-1} + \theta + \gamma) & -\eta_{L-1} & \theta & 0 \\ 0 & 0 & 0 & 0 \\ \sigma & 0 & -\sigma & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \tilde{C}_L = \begin{bmatrix} 0 \\ 0 \\ \eta_{L-1} \\ 0 \end{bmatrix}.$$

The state-transition diagram of $\{\tilde{X}(t), t \geq 0\}$ is shown in Figure 4.

Let $P(t) = (P_0(t), P_1(t), \ldots, P_{L-1}(t), P_L(t))$ be the state probability vector of $\{\tilde{X}(t), t \geq 0\}$, where $P_j(t) = (P_{0,j,0}(t), P_{1,j,0}(t), P_{0,j,1}(t), P_{1,j,1}(t))$ $(j = 0, 1, \ldots, L-1)$ and $P_L = P_{0,L,1}(t)$. The vector $P(t)$ can be obtained by solving the differential matrix equation:

$$\frac{dP(t)}{dt} = P(t).\tilde{Q}, \tag{9}$$

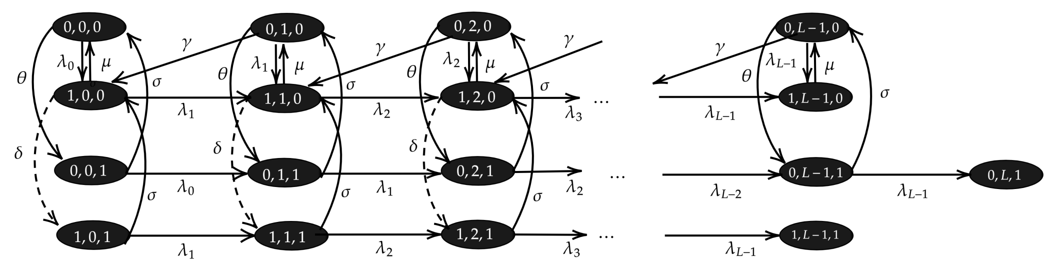


Figure 4: *State-transition diagram of* $\{\tilde{X}(t), t \geq 0\}$

together with $P(t).e = 1$, where $e = (1, 1, \ldots, 1)^t$ is a unit vector with dimension $4L+1$ and $\frac{dP(t)}{dt} = (\frac{dP_0(t)}{dt}, \frac{dP_1(t)}{dt}(t), \ldots, \frac{dP_{L-1}(t)}{dt}, \frac{dP_L(t)}{dt})$, with

$$\frac{dP_j(t)}{dt} = \begin{cases} \left(\frac{dP_{0,j,0}(t)}{dt}, \frac{dP_{1,j,0}(t)}{dt}, \frac{dP_{0,j,1}(t)}{dt}, \frac{dP_{1,j,1}(t)}{dt}\right), & \text{for } j = 0, 1, \ldots, L-1, \\ \frac{dP_{0,L,1}(t)}{dt}, & \text{for } j = L. \end{cases}$$

Applying the Laplace transform to both sides of equation (9), we get:

$$P^*(s)(sI - \tilde{Q}) = P(0), \tag{10}$$

where the initial row vector $P(0)$ is given by $P(0) = (1, 0, 0, 0, 0, \ldots, 0)$ and $P^*(s) = (P_0^*(s), P_1^*(s), \ldots, P_{L-1}^*(s), P_L^*(s))$ with $P_j^*(s) = (P_{0,j,0}^*(s), P_{1,j,0}^*(s), P_{0,j,1}^*(s), P_{1,j,1}^*(s))$ $(j = 0, 1, \ldots, L-1)$, $P_L^* = P_{0,L,1}(s)$ and $P_{i,j,n}^*(s) = \int_0^{+\infty} e^{-st} P_{i,j,n}(t)\, dt$.

Solving equation (10) is a complex task that requires advanced computational techniques. In this study, we leverage MATLAB software to facilitate the solution process. Specifically, by applying the inverse Laplace transform to $P_{1,L-1,0}^*(s)$, $P_{1,L-1,1}^*(s)$ and $P_{0,L,1}^*(s)$, we obtain the expression for $P_{1,L-1,0}(t)$, $P_{1,L-1,1}(t)$ and $P_{0,L,1}(t)$, which gives the probability that the system fails at or before time t, given that the repairer is occupied. The inverse Laplace transform is computed numerically using the algorithm presented below:

Algorithm 1 Laplace Inverse Transform Computation of $P_{1,L-1,0}^*(s)$, $P_{1,L-1,1}^*(s)$, and $P_{0,L,1}^*(s)$

1: Construct the generator matrix $\tilde{Q}$.
2: Compute $q(s) = sI - \tilde{Q}$.
3: Calculate the matrix $A(s) = [sI - Q]^{-1}$ by matrix inversion.
4: Determine the Laplace transform of state probabilities $P^*(s) = P(0) \cdot A(s)$.
5: Calculate the inverse Laplace transforms of $P_{1,L-1,0}^*(s)$, $P_{1,L-1,1}^*(s)$, and $P_{0,L,1}^*(s)$ using the symbolic `ilaplace` function in MATLAB, to obtain the expressions for $P_{1,L-1,0}(t)$, $P_{1,L-1,1}(t)$, and $P_{0,L,1}(t)$.

Let $R(t)$ be the reliability function of the system, then $R(t)$ is given by:

$$R(t) = 1 - P_{1,L-1,0}(t) - P_{1,L-1,1}(t) - P_{0,L,1}(t). \tag{11}$$

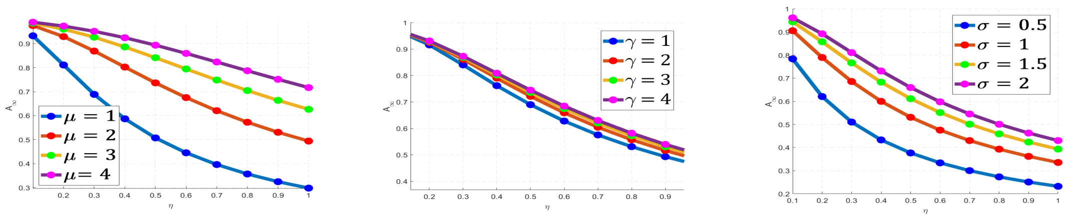
To find the mean time to the first failure ($MTTF$) we use the following formula:

$$MTTF = \int_0^{+\infty} R(t)\, dt. \tag{12}$$

4. Numerical example

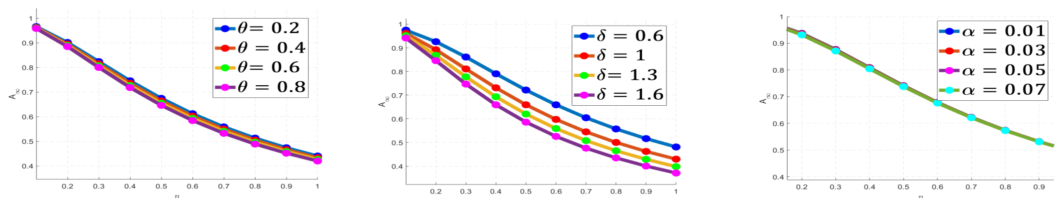
In this section, we study the sensitivity of the system Availability A_∞, reliability $R(t)$ and the mean time to the first failure $MTTF$ with respect to system parameters. This analysis is based on a base case with the following parameter values: $\mu = 2$, $\eta = 0.6$, $\sigma = 1$, $\alpha = 0.05$, $\delta = 0.8$, $\theta = 0.5$ and $\gamma = 3$, $M = 3$, $W = 2$, $C = 1$, $K = 2$. The resulting numerical values for $A(\infty)$ and $R(t)$ are illustrated in figures 5-6 and 7-9, respectively.

Figures 5-9 illustrate that both A_∞ and $R(t)$ exhibit similar behavior with respect to all parameters. Specifically, A_∞ and $R(t)$ increase as μ, γ, and σ rise, while they decrease with higher values of η and δ. The impact of θ is also evident in reducing A_∞, though its effect on $R(t)$ remains negligible. However, the parameter α has little to no influence on either A_∞ or $R(t)$.



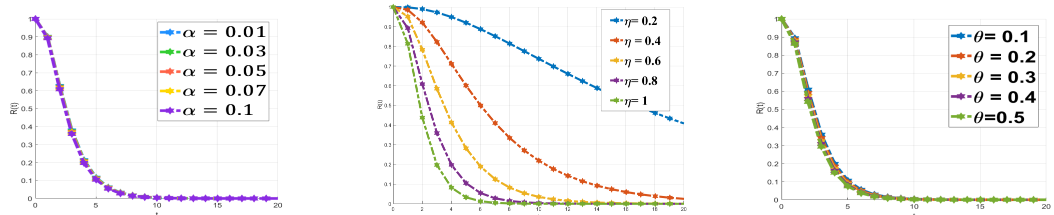
(a) A_∞ vs η for μ (b) A_∞ vs η for γ (c) A_∞ vs η for σ

Figure 5: *Steady-state availability* A_∞ *vs system parameters* μ, γ, *and* σ.



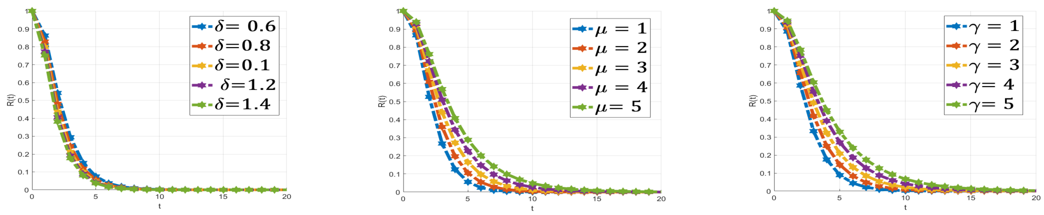
(a) A_∞ vs η for θ (b) A_∞ vs η for δ (c) A_∞ vs η for α

Figure 6: *Steady-state availability* A_∞ *vs system parameters* θ, δ, *and* α.



(a) $R(t)$ vs α (b) $R(t)$ vs η (c) $R(t)$ vs θ

Figure 7: *Reliability function* $R(t)$ *vs system parameters* α, η, *and* θ.



(a) $R(t)$ vs δ (b) $R(t)$ vs μ (c) $R(t)$ vs γ

Figure 8: *Reliability function* $R(t)$ *vs system parameters* δ, μ, *and* γ.

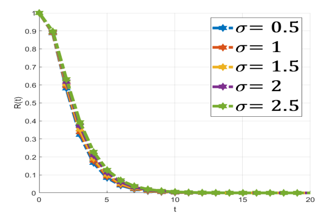


Figure 9: *Reliability function* $R(t)$ *vs system parameter* σ.

Tables 4-9, show the impact of system parameters (α, δ, θ, μ, γ and σ) on the Mean Time to First Failure ($MTTF$) across various values of η.

From Tables 4-9, we observe that when η increases $MTTF$ decreases.

• **MTTF versus** α **and** η: As α increases, $MTTF$ consistently decreases for all values of η. In other words, systems with higher α values are more susceptible to failure, particularly when the system experiences higher arrival rates η.

η	0.1	0.2	0.6	0.8	1
$\alpha = 0.01$	140.1234	31.5767	5.1234	3.2098	2.4895
$\alpha = 0.03$	128.4567	30.2094	5.0672	3.1465	2.4672
$\alpha = 0.05$	118.4785	28.9012	5.0128	3.0869	2.4451
$\alpha = 0.07$	110.3476	27.7215	4.9602	3.0293	2.4234
$\alpha = 0.09$	102.5963	26.5954	4.9085	2.9746	2.4020

Table 4: $MTTF$ *versus* α *and* η

• **MTTF versus** δ **and** η: As δ increases, the $MTTF$ decreases across all values of η. This indicates that higher values of δ result in a shorter mean time to first failure.

η	0.1	0.2	0.6	0.8	1
$\delta = 0.5$	111.5089	28.1384	4.4212	2.8441	2.0396
$\delta = 0.8$	58.5872	18.5609	3.6766	2.4770	1.8323
$\delta = 1$	39.4558	14.1038	3.2244	2.2372	1.6890
$\delta = 1.5$	30.0999	11.6221	2.9228	2.0688	1.5842
$\delta = 2$	24.7013	10.0700	2.7082	1.9443	1.5042

Table 5: $MTTF$ *versus* δ *and* η

• **MTTF versus** θ **and** η: Similar to δ, increasing θ leads to a decrease in $MTTF$.

η	0.1	0.2	0.6	0.8	1
$\theta = 0.2$	167.8474	38.1113	4.8983	3.0468	2.1423
$\theta = 0.4$	111.5089	28.1384	4.4212	2.8441	2.0396
$\theta = 0.6$	80.1144	22.0062	4.0379	2.6730	1.9502
$\theta = 0.8$	62.2545	18.1904	3.7331	2.5294	1.8725
$\theta = 1$	51.2662	15.6842	3.4895	2.4087	1.8050

Table 6: $MTTF$ *versus* θ *and* η

• **MTTF versus μ and η**: Increasing μ, results in a noticeable increase in $MTTF$. Higher μ values enhance system durability and reduce the probability of early failures. This effect is consistent across all stress levels η.

η	0.1	0.2	0.6	0.8	1
$\mu = 1$	33.3047	10.8693	2.4141	1.6881	1.2885
$\mu = 2$	69.9236	19.0933	3.3649	2.2354	1.6440
$\mu = 3$	111.5089	28.1384	4.4212	2.8441	2.0396
$\mu = 4$	153.9382	37.3126	5.5455	3.4970	2.4656
$\mu = 5$	194.8966	46.2035	6.7106	4.1810	2.9147

Table 7: $MTTF$ *versus* μ *and* η

• **MTTF versus γ and η**: Increasing γ significantly boosts $MTTF$. The effect becomes more significant as η increases.

η	0.1	0.2	0.6	0.8	1
$\gamma = 1$	111.5089	28.1384	4.4212	2.8441	2.0396
$\gamma = 2$	198.7491	35.8419	4.6921	2.9510	2.0892
$\gamma = 3$	307.7273	41.2081	4.8511	3.0153	2.1201
$\gamma = 4$	452.1257	45.2651	4.9574	3.0588	2.1414
$\gamma = 5$	655.2260	48.4745	5.0343	3.0904	2.1571

Table 8: $MTTF$ *versus* γ *and* η

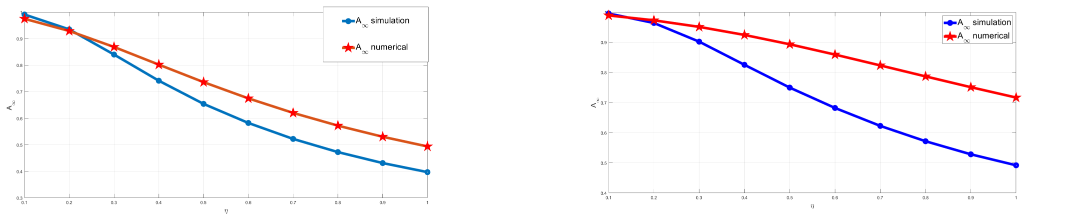
• **MTTF versus σ and η**: As σ increases, $MTTF$ also increases. This effect holds at all values of η.

η	0.1	0.2	0.6	0.8	1
$\sigma = 1$	111.5089	28.1384	4.4212	2.8441	2.0396
$\sigma = 1.5$	174.9496	38.6161	4.8138	2.9923	2.1063
$\sigma = 2$	195.1836	42.9104	5.0091	3.0707	2.1435
$\sigma = 2.5$	202.6555	44.9603	5.1208	3.1177	2.1667
$\sigma = 3$	205.9744	46.0936	5.1916	3.1485	2.1824

Table 9: $MTTF$ *versus* σ *and* η

4.1. Model Validation through Simulation

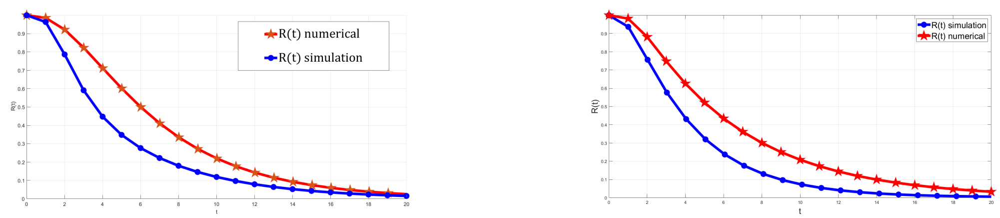
To support the analytical results, we conducted a simulation using TimeNET 5.0.1 and included comparative plots of both the steady-state availability A_∞ and the reliability function $R(t)$ (see Figures 10-11), which confirm the accuracy and consistency of the proposed model.



(a) A_∞ *vs* η *for* $\mu = 2$, $\eta = 0.6$, $\sigma = 1$, $\alpha = 0.05$, $\delta = 0.8$, $\theta = 0.5$, *and* $\gamma = 3$

(b) A_∞ *vs* η *for* $\mu = 4$, $\eta = 0.6$, $\sigma = 1$, $\alpha = 0.05$, $\delta = 0.8$, $\theta = 0.5$, *and* $\gamma = 3$

Figure 10: *Analytical vs simulated steady-state availability* A_∞.



(a) $R(t)$ *vs* t *for* $\mu = 2$, $\eta = 0.6$, $\sigma = 1$, $\alpha = 0.05$, $\delta = 0.8$, $\theta = 0.5$, *and* $\gamma = 3$

(b) $R(t)$ *vs* t *for* $\mu = 4$, $\eta = 0.6$, $\sigma = 1$, $\alpha = 0.05$, $\delta = 0.8$, $\theta = 0.5$, *and* $\gamma = 3$

Figure 11: *Analytical vs simulated reliability function* $R(t)$.

Based on Figures 10-11, the numerical method consistently achieves higher values of $R(t)$ over time, indicating superior reliability performance across the operational period. It also yields smoother reliability curves and higher asymptotic availability $A(\infty)$, reflecting enhanced long-term system dependability. Although simulation methods are advantageous for systems with large state spaces due to their flexibility and computational efficiency, in this case, the numerical approach proves more effective in preserving reliability and delivering robust performance.

5. Conclusion

This paper employed a $GSPN$-based model to assess the reliability and performance of a retrial K-out-of-n system that incorporates cold and warm standbys and an unreliable repairer. A finite-state $CTMC$ is derived from the $GSPN$ model. Then we determined the steady-state availability by resolving the steady-state probability equations in matrix form. Using the Laplace transform method, we obtained the main reliability indices in the transient state. Numerical examples illustrated the sensitivity analysis of reliability indices regarding the parameters of the system. Future work will focus on extending the current model in several key directions. First, we plan to explore Markov Regenerative Stochastic Petri Nets (MRSPNs) to represent systems with general lifetime or repair time distributions. This framework relaxes the exponential time assumption and is expected to yield more accurate evaluations of system reliability and availability, particularly in complex redundant architectures. In addition, we aim to address the commonly made assumption of statistical independence among failures, repairs, and retrials. In many real-world systems, such events are interdependent—failure rates may depend on workload, and repair durations may be correlated. Incorporating these dependencies will enhance the model's realism and applicability to practical scenarios. Finally, future developments will consider weighted k-out-of-n systems, where certain components contribute more critically to overall system function. This extension will enable the model to better reflect scenarios in which components play unequal roles, providing a more nuanced and realistic characterization of system behavior.

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Comparative assessment of flexible manufacturing systems using the EDAS and Shannon entropy method

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Abstract. As global competition intensifies, most manufacturing companies strive to improve their production methods to gain a competitive edge. One such advancement is the adoption of Flexible Manufacturing Systems (FMS), which enable the efficient production of various products in specified quantities with minimal lead times. These systems offer adaptability and efficiency, allowing manufacturers to leverage modern technologies to improve operational performance. However, evaluating or selecting an appropriate FMS involves considering numerous conflicting criteria. To address this complexity, Multi-criteria Decision Making (MCDM) methods are employed. This study conducts a comparative evaluation of eight FMS alternatives using the Evaluation based on the Distance from the Average Solution (EDAS) method, integrated with Shannon Entropy for objective weight determination. Key performance indicators, including production cost, system flexibility, energy efficiency, and operational reliability, are used in the assessment. The Shannon Entropy method ensures unbiased, data-driven weight assignment, while the EDAS method provides a robust framework for ranking alternatives based on their deviation from an average solution. To test the robustness of the ranking, we compared the ranking with other MCDM methods and also conducted a sensitivity analysis using equal weighting criteria. We found that the first and last rankings remained unchanged when we changed the criteria, although there were slight changes in the rankings of some alternatives. The findings highlight the effectiveness of integrating EDAS with Shannon Entropy in selecting the best flexible manufacturing systems, offering valuable insights for manufacturers and decision-makers.

Keywords: EDAS method, flexible manufacturing systems, multi-criteria decision making, normalization, shannon's entropy method

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1. Introduction

In the present time, due to increasing global competition, manufacturing companies need advanced production technologies such as computer-controlled machines, automatic material handling systems, and FMS to quickly meet customer needs and gain a competitive advantage in the global market. With the spread of advanced technology, FMS has attracted the attention of many manufacturing companies, and many companies have switched to this system. The FMS is composed of machine tools and/or robots linked by a transport network and controlled by a central computer to move parts and/or tools as suggested in [17]. The term 'flexible' refers to

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the system's ability to process different types of parts simultaneously at various workstations, with the capability to easily adjust the mix of part types and production quantities in response to changing demand. Potential benefits of using FMS include reducing stock levels, production lead time, installation costs, and providing high flexibility and quality.

The FMSs are systems that can produce various products in the desired quantity in a short time. Therefore, having FMS has become a necessity for many manufacturers. In addition, since the installation of FMS requires a large investment, the selection of the most suitable FMS requires comprehensive analysis and evaluation [5]. Therefore, to get the most benefit from the system in the selection of FMS, a detailed criteria scan must be made, and the importance of these criteria must be determined correctly. If some of these criteria conflict with each other, measurement becomes difficult, and a balance must be established between these criteria when considering the preferences of the decision-makers in the production enterprise. Since many criteria are included in the selection phase of FMS, the MCDM method has been used many times in the literature on FMS selection.

Some of the MCDM methods used for FMS selection in the literature are as follows. The weight of the criteria can be calculated by the AHP method [4, 2]. Another axiomatic design method is discussed in the literature [19], where the author used the Entropy method to determine the criterion weights, and they call preference ranking methods such as EVAMIX (Evaluation of Mixed Data), COPRAS (Complex Proportional Assessment), ORESTE (Organization, Rangement Et Synthese De Donnes Relationnelles), OCRA (Operational Competitiveness Rating), and EDAS in ranking the FMSs. The DEMATEL (Decision-Making Trial and Evaluation Laboratory) method was used to find the relationship density between the criteria and calculate the weight of the criteria. The DEMATEL method evaluated decisions about whether or not to install FMS. Some study evaluated the flexible production systems by combining heuristic fuzzy logic and multi-attribute group decision-making methods [13].

2. Literature review

The discipline of production management is relevant to this study. As the study [16] state that the core components of field material handling systems include computer numerical control machine tools, which are loaded and unloaded, along with automated material handling devices by advanced industrial robots, computer-controlled storage and retrieval systems, and other automated equipment. According to [6], FMS issues are divided into four categories: design, planning, scheduling, and control. The right amount of each type of machine tool, the capacity handling system of the material, and the buffer size are among the design issues with FMSs. Planning issues in FMSs include choosing which parts to process at the same time, grouping machine tools optimally, assigning operations to different components, and assigning pallets and fixtures to different part types. Finding the ideal order for machine tools and components as input sequences is one of the scheduling issues associated with FMSs. Monitoring the system to make sure that requirements and deadlines are followed and that unreliability issues are taken into account is known as FMS control problems [22, 6]. The design challenge is the main topic of the study that is being suggested in this paper.

Prior research has concentrated on machine flexibility and routing, which affect several performance metrics. Productivity enhancement, machine selection, number allotted, capacity, buffer sizes, pallet allocations, material handling systems, jigs and fixtures allocations, FMS planning, scheduling, optimization of limited resources, and FMS controls are among the issues associated with FMSs [20]. The increasing complexity and competitiveness of global manufacturing systems have prompted researchers to adopt MCDM approaches for evaluating flexible and sustainable production alternatives. Together, these studies demonstrate the adaptability and effectiveness of MCDM methods across a variety of domains, including trade, cybersecurity, sustainable manufacturing, and workforce development. Their collective insights support the use of MCDM—particularly integrated approaches like EDAS with Shannon Entropy—in the comparative evaluation of FMS under complex and dynamic decision environments.

Based on the aforementioned investigations, researchers have identified several elements that have a major influence on the performance of FMSs. The researchers include a variety of topics, including the need for design modifications in the final product, cutting circumstances, worker skill and adaptability, sequencing flexibility, routing flexibility, part sequencing, and determining the maximum number of routes. MCDM techniques, such as Fuzzy COPRAS or EDAS, have been used in several studies to assess the impact of compelling factors and variables on the performance of flexible manufacturing systems. Furthermore, to find the ideal value of variables and improve the overall performance of flexible manufacturing systems, further research has concentrated on optimization or simulation-based optimization techniques. Statistical analysis is the main tool which are utilized by other researchers, such as [3], to evaluate the impact of various variables and factors on FMS performance. By using a hybrid framework that combines MCDM, Entropy, and EDAS simulation, the suggested framework achieves all of these goals. The stated goals and possible capabilities are demonstrated by applying this technique to an actual industrial case study.

This paper introduces a very elaborate framework of decision-making involving the validation and prioritizing of Flexible Manufacturing Systems by unearthing the EDAS procedure with the Shannon Entropy technique to find objective weights. The Shannon Entropy approach generates a data-driven, step-by-step process by which it becomes possible to determine the relative importance of criteria through the use of inherent information variability within the data set. With the combination of EDAS, which assesses the alternatives based on positive and negative distances to an average solution, the suggested methodology can provide a strong framework to assess FMS that is transparent and objective. The main value of the current work is the use of this hybrid method on real FMS data and proving its practical value and its computational efficiency. The method will allow decision-makers to deal with multi-criteria analysis where the economic and technical aspects of the projects conflict (e.g., labor cost, quality improvement, floor area, capital cost). To show the robustness of the proposed EDAS Entropy framework, we compared it with existing MCDM methods such as COPRAS and the SWEI (Sum Weghted Exponential Information) MCDM methods. The similarity of orderings of the methods proves the reliability and soundness of the suggested method. Some of the major strengths are ease of implementation, lower computational complexities than pairwise-based methods, objective weight assignment, and stability or high ranking, irrespective of the weighting schemes employed. The above advantages turn the given approach into a useful instrument to be used to address the issue of selecting FMS and other complicated decision-making tasks in the manufacturing and industrial fields.

This study employs the entropy and EDAS method for selecting an FMS. Its originality lies in the fact that the entropy and EDAS methods have not previously been applied to FMS selection. Given that the decision matrix was already known, the Entropy method was utilized to determine the objective weights. The CRITIC (CRiteria Importance Through Inter-criteria Correlation) method, which is another method of finding objective weights, was not preferred in this study because it requires more processing. In MCDMs such as PROMETHEE (Preference Ranking Organization Method for Enrichment Evaluation) and ELECTRE (Elimination and Choice Translating Reality), increasing the number of alternatives increases the processing time. Even though the number of alternatives increases in the ROV (Range of value) method, the processing time is very short compared to these methods since pairwise comparison is not made. Therefore, the EDAS method was preferred in this study.

3. Research methodology

The study methodology's goal is to offer an organized and methodical process for choosing the best FMS by utilizing the entropy and EDAS techniques. The methodology seeks to integrate both qualitative and quantitative criteria to ensure a comprehensive evaluation of FMS alternatives. By leveraging entropy for determining objective weights, it ensures impartiality in assigning importance to each criterion, minimizing biases from subjective judgments. The primary goal is to enhance decision-making accuracy in FMS selection by addressing multiple conflicting factors, such as cost, efficiency, flexibility, and technological compatibility. Additionally, the methodology aims to demonstrate the effectiveness of the entropy-based EDAS method in real-world applications, validating its utility in complex industrial environments. This approach provides a replicable and transparent framework for decision-makers in manufacturing industries to optimize their systems and improve productivity. Ultimately, it seeks to fill the gap in existing research by applying the EDAS MCDM method for FMS selection. Finally, COPRAS and SWEI MCDM methods are used to check the robustness of the ranking.

3.1. The Shannon's entropy method

To enhance objectivity in the evaluation process, objective weighting methods such as Shannon entropy are often integrated with MCDM techniques. The Shannon entropy method assigns weights to decision criteria based on the degree of variability in the data, reducing human bias and ensuring a data-driven evaluation. Information uncertainty is quantified using the idea of entropy. Information entropy [15] is a measure of how disordered a system is and how much valuable information can be extracted from the available data. Numerous papers have employed the weighted entropy method to resolve MCDM issues. The combined PROMETHEE and Entropy methods is applied for supplier selection [25]. The entropy method is used for calculating the criterion weight in many studies [7, 8, 9, 10, 11, 12]. Another study [1], the Entropy and VIKOR (VIekriterijumsko KOmpromisno Rangiranje) methods is used for sustainable strategy selection for SMEs.

The entropy method can be summarized in 5 steps [21]:

Step 1: As the first stage, a decision matrix (DM) containing criteria and alternatives is created. Equation (1) shows the decision matrix:

$$DM_{i,j} = [a_{i,j}]_{m\times n} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}. \tag{1}$$

In the equation (1), $a_{i,j}$ represents the performance in the criterion. This decision matrix includes a total of $j = 1, 2, 3, ...n$ criteria and $i = 1, 2, 3, ...m$ alternatives.

Step 2: The values in the decision matrix are standardized using equation (2) for the beneficial criteria, and equation (3) for the non-beneficial criteria. $r_{i,j}$ values in the equations shows the standardized version of the $a_{i,j}$ value.

$$r_{i,j} = \frac{a_{i,j}}{\max a_{i,j}}, \tag{2}$$

$$r_{i,j} = \frac{\min a_{i,j}}{a_{i,j}}. \tag{3}$$

Step 3: Standardized values are normalized using equation (4), where $s_{i,j}$ shows the normalized value.

$$s_{i,j} = \frac{r_{i,j}}{\sum_{i=1}^{m} r_{i,j}}. \tag{4}$$

Step 4: After the normalization process, the entropy value of each criterion is obtained using equation (5). The value of H_j represents the entropy of the j criterion.

$$H_j = -\frac{\sum_{j=1}^{n} s_{i,j} \log_2 s_{i,j}}{\log_2 m}. \tag{5}$$

Step 5: In the last step, the weight of each criterion is obtained.

$$w_j = \frac{1 - H_j}{\sum_{j=1}^{n} 1 - H_j}. \tag{6}$$

The value of w_j in equation (6) represents the objective weight of the criterion is represented. Through the application of the entropy method, these objective criterion weights are determined. After the last step of this method, the EDAS method is used, and the objective weights found by this method are transferred to rank the alternatives.

3.2. The EDAS method

The EDAS is an MCDM method used to assess and rank alternatives across various criteria. This method assesses how close or far an alternative is from an average solution, considering both positive and negative distances from the average. Evaluating and selecting appropriate FMS alternatives is inherently complex, involving multiple, often conflicting criteria such as cost, flexibility, reliability, and energy efficiency. To address this complexity, MCDM methods have been increasingly employed in manufacturing system evaluations. Among these methods, the EDAS approach has gained attention due to its simplicity, computational efficiency, and ability to consider both positive and negative deviations from an average solution [14]. This method evaluates and selects the suppliers based on criteria such as quality, cost, and delivery time. This prioritizes projects or tasks considering factors like cost, benefit, risk, and time. Distributing resources effectively among competing projects or departments. Assessing policy options based on social, economic, and ecological criteria. Choosing the best product design considering multiple factors like cost, functionality, and customer preference.

The EDAS method is a powerful tool for decision-makers dealing with complex scenarios involving multiple criteria. Its ability to provide a balanced evaluation of alternatives, simplicity, and robustness make it a popular choice in various fields. The EDAS technique was utilized to compute the performance of alternatives and then rank the alternatives. The EDAS technique consists the following steps [26].

Step 1: In this step, determining the average solutions of the matrix (AV_j) is formed by taking the average solutions of the criteria.

$$AV = [AV_j]_{1\times m}. \tag{7}$$

Where AV_j represents the average of criterion j and is calculated as:

$$AV_j = \frac{\sum_{i=1}^{n} a_{i,j}}{n}. \tag{8}$$

Step 2: For each criterion, calculate the positive distance from average (PDA) as well as the negative distance from average (NDA) from the average by using (9) and (10), and then matrices are obtained. Based on whether the objective is benefit-based or cost-based, various equations are used to compute each element of the generated matrices ($PDA_{i,j}, NDA_{i,j}$).

$$PDA = [PDA_{i,j}]_{n\times m}, \tag{9}$$

$$NDA = [NDA_{i,j}]_{n\times m}. \tag{10}$$

If j-th criterion is beneficial, the values of $PDA_{i,j}$ and $NDA_{i,j}$ are calculated as:

$$PDA_{i,j} = \frac{\max(0, (a_{i,j} - AV_j))}{AV_j}, \tag{11}$$

$$NDA_{i,j} = \frac{\max(0, (AV_j - a_{i,j}))}{AV_j}. \tag{12}$$

Similarly, if j-th criterion is non-beneficial, the values of $PDA_{i,j}$ and $NDA_{i,j}$ are calculates as:

$$PDA_{i,j} = \frac{\max(0, (AV_j - a_{i,j}))}{AV_j}, \tag{13}$$

$$NDA_{i,j} = \frac{\max(0, (a_{i,j} - AV_j))}{AV_j}. \tag{14}$$

Step 3: With the help of the following equations (15) and (16) the weighted positive value (SP_i) and weighted negative values (SN_i) are calculated respectively. The equation is multiplied by the weight of the criteria computed by using Shannon's entropy method, i.e.

$$SP_i = \sum_{j=1}^{n} w_j PDA_{i,j}, \tag{15}$$

$$SN_i = \sum_{j=1}^{n} w_j NDA_{i,j}. \tag{16}$$

Step 4: In this step, the weighted positive value of SP_i and weighted negative value SN_i are normalized as:

$$NSP_i = \frac{SP_i}{\max_i(SP_i)}, \tag{17}$$

$$NSN_i = 1 - \frac{SN_i}{\max_i(SN_i)}. \tag{18}$$

Step 5: Finally, for each alternative, the appraisal scores (AS_i) are evaluated as follows:

$$AS_i = \frac{1}{2}(NSP_i + NSN_i). \tag{19}$$

Alternative appraisal score (AS_i) lies in the range $0 \leq AS_i \leq 1$, where the higher the value, the closer the alternative is to the ideal (average-based) choice.

3.3. The SWEI method

The SWEI MCDM was developed by [8]. It is very useful in the decision-making domain, i.e. there are many theoretical mplications of SWEI in the MCDM. The model enhances the accuracy and robustness of decision-making processes by effectively handling criterion interdependencies and uncertainties. The SWEI offers significant advantages in modelling interdependencies and handling uncertainty; it is essential to consider the potential complexity it introduces to MCDM models. The integration of SWEI requires advanced computational techniques and a deep understanding of the underlying decision-making context, which results in its effective application in solving complex problems. This complexity must be managed to prevent it from becoming a barrier to practical implementation. The fundamental principle of the SWEI method is that alternatives with higher information content receive lower preference rankings, whereas those with lower information content are assigned higher ranks. This is based on the inverse relationship between the probability of an alternative and its associated information content—fewer probable alternatives (i.e., more uncertain or diverse in performance) convey greater informational entropy, thus indicating less desirability in the decision-making context. The following are the steps involved in SWEI MCDM [7, 8]:

Step 1: Construct the information decision matrix $IDM = [a_{i,j}]_{m\times n}$ according to (20), where $a_{i,j} > 0$:

$$IDM_{i,j} = [a_{i,j}]_{m\times n} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}. \tag{20}$$

Step 2: Normalize the decision matrix for beneficial and non-beneficial criteria according to (21) and (22):

$$\overline{IDM}_{i,j} = \frac{a_{i,j}}{\sum_{i=1}^{m} a_{i,j}}, \quad \text{beneficial criteria,} \tag{21}$$

$$\overline{IDM}_{i,j} = \frac{1/a_{i,j}}{\sum_{i=1}^{m} 1/a_{i,j}}, \quad \text{non-beneficial criteria.} \tag{22}$$

Step 3: Compute the information score $(IS_{i,j})$ of the attributes of the normalized decision matrix for the alternatives according to (23):

$$IS_{i,j} = \log_2\left(\frac{1}{\overline{IDM}_{i,j}}\right). \tag{23}$$

Step 4: Compute the weighted exponential information score of the decision matrix (IDM_i'') for the alternatives according to (24):

$$SWEI = IDM_i'' = \sum_{j=1}^{n}\left(\log_2\left(\frac{1}{\overline{IDM}_{i,j}}\right)\right)^{w_j}, \tag{24}$$

where w_j is the weight of the criterion such that $\sum_{j=1}^{n} w_j = 1$.

Step 5: Finally, the summing the weighted exponential information scores for all alternatives and sorted in ascending order to establish the final preference ranking. The alternative with the lowest score is designated the first rank.

3.4. The COPRAS method

The COPRAS MCDM was introduced by [23]. It stands out as an MCDM technique used to analyze and rank a series of alternatives based on a set of conflicting criteria. It is especially useful where there is a need to both maximize and minimize. The COPRAS enables us to evaluate the alternatives with direct proportional measurement based on the criteria's meanings (weights) and ideas of how each alternative performs compared with the ideal one. A process entails the normalization of the decision matrix, the use of weightings, and the calculation of both the sums of beneficial (maximizing) and non-beneficial (minimizing) criteria of different alternatives. The following are the steps involved in COPRAS:

Step 1: Construct the decision matrix $DM = [a_{i,j}]_{m\times n}$ according to (20)

Step 2: Normalize the decision matrix according to (25):

$$s_{i,j} = \frac{a_{i,j}}{\sum_{i=1}^{m} a_{i,j}}. \tag{25}$$

Step 3: Compute the weighted normalized decision matrix for the alternatives using:

$$r_{i,j} = s_{i,j} \times w_j, \tag{26}$$

where w_j is the weight of the criterion and $\sum_{j=1}^{n} w_j = 1$.

Step 4: Compute the sums of weighted normalized values for each alternative for beneficial (maximizing) criteria and non-beneficial (minimizing) criteria by (27) and (28), respectively.

$$S_i^+ = \sum_{j=1}^{n} r_{i,j}, \quad \text{beneficial criteria}, \tag{27}$$

$$S_i^- = \sum_{j=1}^{n} r_{i,j}, \quad \text{non-beneficial criteria}. \tag{28}$$

Step 5: Compute the relative significance for the alternatives according to (29):

$$Q_i = S_i^+ + \frac{\min S_i^- \sum_{i=1}^{m} S_i^-}{S_i^- \sum_{i=1}^{m} \left(\frac{\min S_i^-}{S_i^-} \right)}. \tag{29}$$

Step 6: Compute the utility degree (%) for the alternatives according to (30):

$$U_i = \frac{Q_i}{\max Q_i} \times 100\%. \tag{30}$$

The alternative with the highest utility degree U_i is considered the most preferable. COPRAS is valued for its simplicity, transparency, and capacity to handle conflicting objectives in decision-making processes.

4. Results and discussion

FMS is an innovative manufacturing system that manage and create a wide range of products with minimum operator interaction. It combines automated material handling systems, centralized control systems, and computer-controlled equipment to achieve high levels of flexibility and productivity. FMS allows for the production of different models and variants of the same production line without significant downtime or reconfiguration. It ensures components and subassemblies are produced in sync with demand, reducing inventory costs and improving efficiency. It is capable of producing various electronic products, such as smartphones and laptops, which require frequent design changes and updates. Industries with high customization, quick adaptability, and efficient production traditionally use FMS. FMS system is particularly useful in automotive manufacturing, where multiple variants and parts have to be assembled on the same assembly line. This leads to seamless changes between design products without downtime, therefore increasing productivity and minimizing costs. FMS is used in electronics to support the rapid creation of circuit boards and components to meet evolving consumer requirements. Aerospace industries make use of FMS for manufacturing complex parts with precision, which helps ensure consistency and quality are assured.

4.1. Case study description

As industries worldwide adopt Industry 4.0 practices, research in FMS ensures that manufacturers stay competitive by leveraging the latest advancements in automation and digitalization. The FMS is enhancing the resilience of manufacturing systems to respond to global supply chain disruptions, fluctuating demands, and unforeseen challenges. In summary, FMS plays a crucial role in modern manufacturing, and ongoing research is essential to harness its full potential, ensuring adaptability, efficiency, and competitiveness in a rapidly evolving industrial landscape. In this study, the data is taken from [24] for the selection problem of an FMS. They identified seven criteria with eight alternatives for the FMS problem. These seven criteria are as follows; reduction in labor cost (C-1), the percentage reduction in installation cost (C-2), the percentage reduction in the amount of backlog between process steps (C-3), increase in market response (C-4), increase in quality (C-5), floor area in square meters (C-6), capital and maintenance cost in thousand dollars (C-7). Of these seven criteria, the first five criteria are beneficial (max), and the last two criteria are non-beneficial (min) criteria. The decision matrix is tabulated in Table 1.

Criteria →	C-1	C-2	C-3	C-4	C-5	C-6	C-7
Alternatives ↓	max	max	max	max	max	min	min
A-1	30	5	23	0.745	0.745	5000	1500
A-2	18	15	13	0.745	0.745	6000	1300
A-3	15	10	12	0.5	0.5	7000	950
A-4	25	13	20	0.745	0.745	4000	1200
A-5	14	14	18	0.255	0.745	3500	950
A-6	17	9	15	0.745	0.5	5250	1250
A-7	23	20	18	0.5	0.745	3000	1100
A-8	16	14	8	0.255	0.5	3000	1500

Table 1: *Decision matrix for selection of the best FMS.*

The decision matrix is standardized by applying the first step of the Entropy method from eq. (2) and (3) and is given in Table 2. The standardized decision matrix is normalized using eq. (4) an is given in Table 3.

Alternatives	C-1	C-2	C-3	C-4	C-5	C-6	C-7
A-1	1.000	0.250	1.000	1.000	1.000	0.600	0.633
A-2	0.600	0.750	0.565	1.000	1.000	0.500	0.731
A-3	0.500	0.500	0.522	0.671	0.671	0.429	1.000
A-4	0.833	0.650	0.870	1.000	1.000	0.750	0.792
A-5	0.467	0.700	0.783	0.342	1.000	0.857	1.000
A-6	0.567	0.450	0.652	1.000	0.671	0.571	0.760
A-7	0.767	1.000	0.783	0.671	1.000	1.000	0.864
A-8	0.533	0.700	0.348	0.342	0.671	1.000	0.633

Table 2: *Standardized decision matrix for the FMS.*

4.2. The EDAS is integrated with the Shannon entropy method

The entropy of each criterion is calculated using eq. (5), after normalization. The criterion weights are calculated using eq. (6). The weights of the criterion computed by the Entropy method are tabulated in the last row of Table 3. A graphical illustration of the criterion weight is given in Figure 1. The criteria are listed as follows according to their weights: $C-4 > C-2 > C-6 > C-3 > C-1 > C-5 > C-7$. The most significant criterion is C-4 (market response). After calculating the weights of the criteria, the EDAS method is used. In the first step, the average value (AV_j) of the decision matrix from Table 1 is determined. The average value is formed by taking the average solutions of the criteria in the normalization process. Next, the PDA and NDA are calculated from Table 3 by using eq. (9) and (10). Tables 4 and 5 show the values of PDM and NDM, respectively.

Alternatives	C-1	C-2	C-3	C-4	C-5	C-6	C-7
A-1	0.190	0.050	0.181	0.166	0.143	0.105	0.099
A-2	0.114	0.150	0.102	0.166	0.143	0.088	0.114
A-3	0.095	0.100	0.094	0.111	0.096	0.075	0.156
A-4	0.158	0.130	0.157	0.166	0.143	0.131	0.123
A-5	0.089	0.140	0.142	0.057	0.143	0.150	0.156
A-6	0.108	0.090	0.118	0.166	0.096	0.100	0.119
A-7	0.146	0.200	0.142	0.111	0.143	0.175	0.135
A-8	0.101	0.140	0.063	0.057	0.096	0.175	0.099
Average	19.8	12.5	15.9	0.56	0.65	4594	1219
Weights	0.121	0.213	0.149	0.256	0.060	0.150	0.050

Table 3: *Normalized decision matrix for the FMS.*

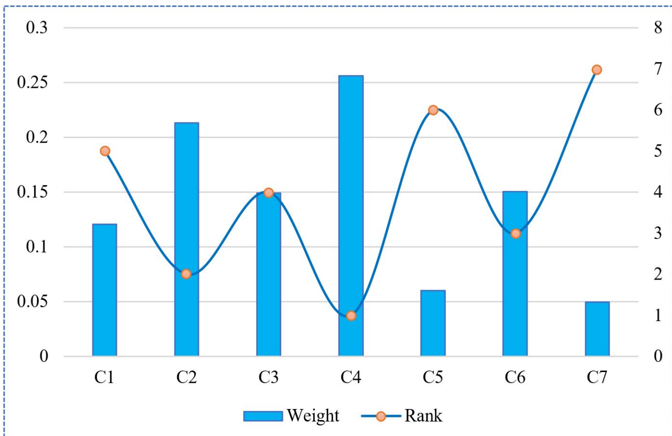

Figure 1: *Criterion weight score with ranking.*

Alternatives	C-1	C-2	C-3	C-4	C-5	C-6	C-7
A-1	0.519	0.000	0.449	0.327	0.141	0.000	0.000
A-2	0.000	0.200	0.000	0.327	0.141	0.000	0.000
A-3	0.000	0.000	0.000	0.000	0.000	0.000	0.221
A-4	0.266	0.040	0.260	0.327	0.141	0.129	0.015
A-5	0.000	0.120	0.134	0.000	0.141	0.238	0.221
A-6	0.000	0.000	0.000	0.327	0.000	0.000	0.000
A-7	0.165	0.600	0.134	0.000	0.141	0.347	0.097
A-8	0.000	0.120	0.000	0.000	0.000	0.347	0.000

Table 4: *Positive decision matrix from the average value.*

Finally, weighted positive values (SP_i), weighted negative values (SN_i), weighted positive normalized value (NSP_i), weighted negative normalized value (NSN_i), and appraisal scores (AS_i) are computed. Weighted normalized positive value represents the weighted contribution of how well an alternative performs above average. Weighted normalized negative values represent the weighted contribution of how poorly an alternative performs below average. Their graphical representation is shown in Figure 2. The final ranking of flexible manufacturing systems is shown in Table 6. Figure 3 illustrates the FMSs and their final ranking based on Entropy and EDAS techniques. Based on the results of the evaluation of the EDAS approach, the best

FMS for the particular application of the industrial work under the current parameters is A-7 followed by A-4, A-1, A-2, A-6, A-5, A-8 and A-3.

Alternatives	C-1	C-2	C-3	C-4	C-5	C-6	C-7
A-1	0.000	0.600	0.000	0.000	0.000	0.088	0.231
A-2	0.089	0.000	0.181	0.000	0.000	0.306	0.067
A-3	0.241	0.200	0.244	0.109	0.234	0.524	0.000
A-4	0.000	0.000	0.000	0.000	0.000	0.000	0.000
A-5	0.291	0.000	0.000	0.546	0.000	0.000	0.000
A-6	0.139	0.280	0.055	0.000	0.234	0.143	0.026
A-7	0.000	0.000	0.000	0.109	0.000	0.000	0.000
A-8	0.190	0.000	0.496	0.546	0.234	0.000	0.231

Table 5: *Negative decision matrix from the average value.*

Alternatives	SPi	SNi	NSPi	NSNi	ASi	Rank
A-1	0.222	0.153	0.951	0.418	0.685	3
A-2	0.135	0.087	0.578	0.668	0.623	4
A-3	0.011	0.229	0.047	0.127	0.087	8
A-4	0.192	0.000	0.823	1.000	0.911	2
A-5	0.101	0.175	0.432	0.334	0.383	6
A-6	0.084	0.122	0.359	0.536	0.448	5
A-7	0.233	0.028	1.000	0.893	0.947	1
A-8	0.078	0.262	0.333	0.000	0.167	7

Table 6: *Final ranking of the FMS by the EDAS method.*

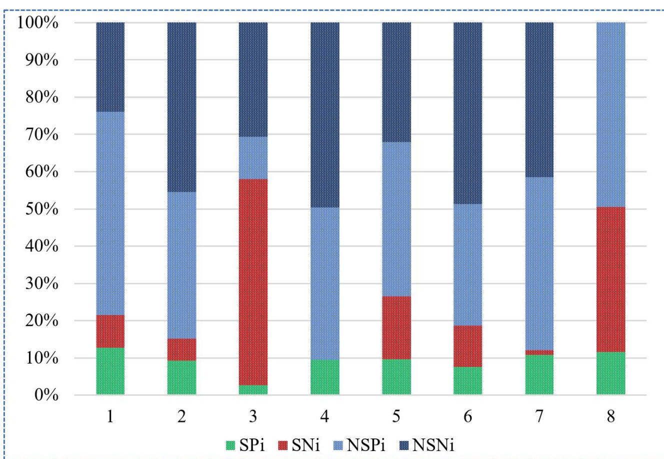

Figure 2: *Graphical representation of* SP_i, SN_i, NSP_i *and* NSN_i.

These outcomes are similar to those proposed by [24] with the fuzzy programming technique for multiple objectives. It should be noted, nevertheless, that the user's assessments of relative relevance determine the ranking. If the user gives the characteristics varying relative priority values, the ranking that is shown could be altered. This method is equivalent to the study [18]. The underlying concept is that criteria with higher variability (or more "entropy") contain more information and should be given more weight. In contrast, the approach presented in this work offers a clear-cut, rational, and basic process for solving the FMS selection problem.

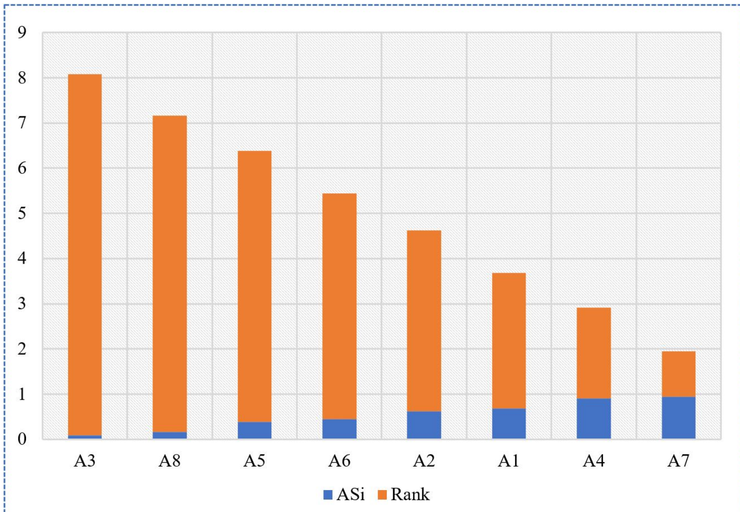

Figure 3: *Illustration of FMS ranking with the scores.*

4.3. Comparison with the SWEI and COPRAS

We further compare the robustness of the ranking generated by the EDAS method with the SWEI and COPRAS methods. Firstly, the decision matrix is constructed for SWEI and COPRAS according to eq. (20), in the first step from the raw data shown in Table 1. In the second step, the data is normalized using eq. (21) and (22) for the beneficial and non-beneficial criteria for SWEI and for COPRAS using eq. (24). The normalized values are tabulated in Table 7 for SWEI.

Alternatives	C-1	C-2	C-3	C-4	C-5	C-6	C-7
A-1	0.190	0.050	0.181	0.166	0.143	0.105	0.099
A-2	0.114	0.150	0.102	0.166	0.143	0.088	0.114
A-3	0.095	0.100	0.094	0.111	0.096	0.075	0.156
A-4	0.158	0.130	0.157	0.166	0.143	0.131	0.123
A-5	0.089	0.140	0.142	0.057	0.143	0.150	0.156
A-6	0.108	0.090	0.118	0.166	0.096	0.100	0.119
A-7	0.146	0.200	0.142	0.111	0.143	0.175	0.135
A-8	0.101	0.140	0.063	0.057	0.096	0.175	0.099

Table 7: *Normalized decision matrix by the SWEI for the FMS.*

In the third step eq. (23) is applied to rank the alternatives. Finally, in the fourth step, the alternatives are ranked in ascending order. Secondly, for the COPRAS method, the weighted normalized matrix is calculated by eq. (25), then the weighted normalized values are calculated by eq. (26) and (27) for the beneficial and the non-beneficial criteria, respectively. After calculating the relative importance for the alternatives by eq. (28), the utility degree is calculated by eq. (29). Afterwards, the alternative with the highest utility degree U_i is considered the most preferable. The ranking is shown in Table 8, which is generated using entropy weights. Table 8 also shows the ranking comparison of EDAS, COPRAS, and SWEI with the entropy weighting method. Figure 4 provides a comparison of the ranking results. Alternative A-7 consistently achieved the 1st rank across all evaluation approaches. Alternatives A-4 secured the 2nd position, under all methods. Alternative A-1 was ranked in 3rd position by EDAS and COPRAS method, while it got 4th position by the SWEI method. Similarly, Alternative A-2 was ranked in 4th position by EDAS and COPRAS method, while it got 3rd position by the SWEI method. The alternative A-6 occupied the 5th position in all three methods. Similarly, Alternative A-5 was placed 6th by EDAS, COPRAS, and SWEI method under the entropy weighting method. Alternative A-8 was assigned the 7th position under the EDAS, COPRAS, and SWEI methods with entropy weights. Finally, alternative A-3 consistently received the last rank across all methods, regardless of the weighting strategy.

Alternatives	Appraisal scores by EDAS	Rank	Information scores by SWEI	Rank	Utility degree by COPRAS	Rank
A-1	0.685	3	8.218	4	87.55	3
A-2	0.623	4	8.189	3	86.44	4
A-3	0.087	8	8.359	8	65.19	8
A-4	0.911	2	8.114	2	97.44	2
A-5	0.383	6	8.296	6	76.37	6
A-6	0.448	5	8.247	5	78.69	5
A-7	0.947	1	8.106	1	100	1
A-8	0.167	7	8.341	7	66.99	7

Table 8: *Ranking comparison scores of the EDAS with other MCDM methods.*

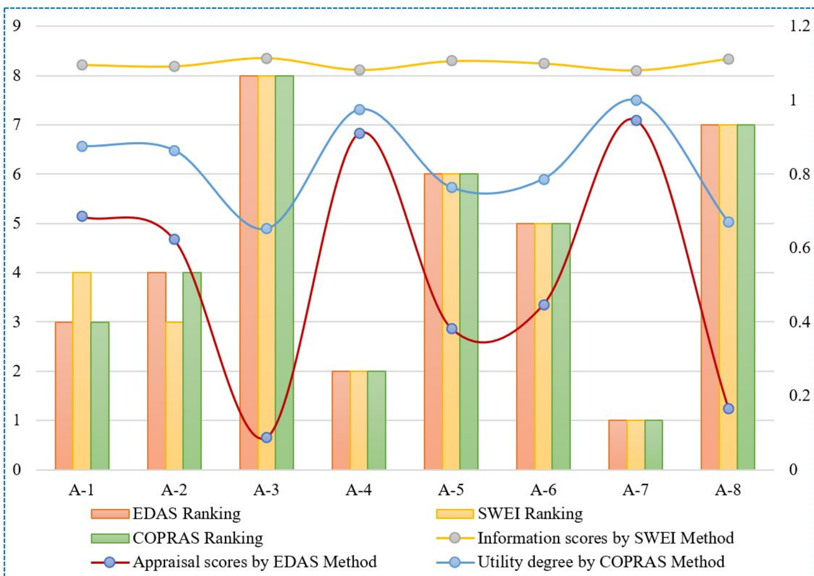

Figure 4: *Comparison of the ranking of the EDAS method with the SWEI and COPRAS methods.*

4.4. Sensitivity analysis

This subsection assesses the sensitivity of the ranking to variations in the input parameters or criteria weights. By systematically adjusting these values within a reasonable range and observing corresponding changes in the rankings, we can gain insights into the stability and robustness of the decision-making process. This analysis helps identify which criteria or parameters exert the most significant influence on the final rankings and how variations in these factors affect the relative positions of alternatives. Additionally, examining the impacts of different values allows us to understand the trade-offs inherent in the decision-making process and explore potential scenarios under varying conditions. By conducting this sensitivity analysis, decision-makers can better understand the implications of their choices and make more informed decisions that account for uncertainties and variations in input parameters. Table 9 shows the ranking by applying the equal weight for seven criteria (1/7 = 0.143) and entropy weight method. All individual rankings and the final ranking are the same. Nevertheless, a few slight ranking modifications compared to Table 8 can be observed. Figure 5 shows the ranking difference between the entropy weight and the equal weight method. As the weights changed, the ranking of A-7 and A-3 remained practically the same. So, the proposed EDAS method delivers the same results for decisions even with changed weighting scenarios, which is necessary for real-world use. It points out the factors that play the most important role in the end rankings. It allows those in charge to determine which factors are the most important and pay attention to these in future policy adjustments or assessments. Since there is little difference in the ranks between the two weight methods, it is clear that the approach stays consistent and flexible in various decision-making situations. The comparison between the entropy-based (objective) and equal (neutral) approaches proves that the method performs well in both precise and simple circumstances for decision-making.

Alternative	EDAS score by entropy weight	Ranking by entropy weight	EDAS score by equal weight	Ranking by equal weight
A1	0.685	3	0.71	3
A2	0.623	4	0.54	5
A3	0.087	8	0.12	8
A4	0.911	2	0.90	2
A5	0.383	6	0.54	4
A6	0.448	5	0.35	6
A7	0.947	1	0.97	1
A8	0.167	7	0.16	7

Table 9: *Sensitivity analysis by different weights for the FMS.*

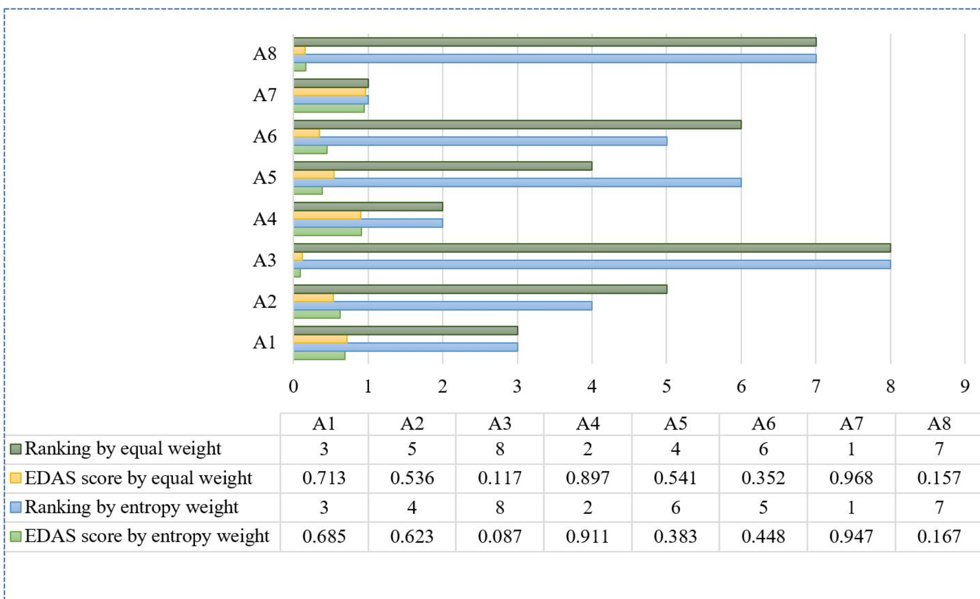

Figure 5: *Sensitivity analysis by different weights for the FMS.*

Worldwide manufacturing rivalry has compelled businesses to think about cutting costs while boosting quality and customization response. The development of FMSs promises enormous possibilities for enhancing flexibility and altering the competitive landscape by simultaneously guaranteeing cost-effectiveness and customized manufacture. Before investing in such innovative manufacturing methods that demands large capital outlays, one must take into account both the impacts that are easily quantified in dollars and those that are not. While precision-based approaches have been previously suggested to tackle the assessment and choice of flexible manufacturing technologies, these approaches are complex, knowledge-intensive, and may be beyond the reach of the individual decision-maker or user organization. A straightforward and rational scientific approach or mathematical instrument is required to assist user organizations in making an appropriate decision. This work proposes a methodology for choosing an appropriate FMS from a wide range of alternative FMSs. The methodology is based on a combinatorial mathematical approach combined with Shannon's entropy. When compared to other approaches that have been suggested in the past for choosing an FMS, this strategy is easier to put into practice. The FMSs selection characteristics function created in this work aids in the creation of the FMSs selection index, which assesses and ranks FMSs for a particular selection problem and may concurrently take into account any number of both quantitative and qualitative FMS selection attributes. In addition, the article proposes an entropy and EDAS conversion scale with ranked value judgments to express the qualitative FMS selection feature. The suggested technique is unique in that it provides a generic process that may be used for a variety of management science selection challenges that involve ambiguity and several both qualitative and quantitative selection qualities. Practical issues could come up when this kind of technique is put into practice. It is necessary to ascertain the values corresponding to the qualities of several possible FMS that the suggested technique would assess. These figures might occasionally be estimates. A what-if study, including many runs of the suggested model, can be necessary due to a lack of precise and trustworthy data. What-if analysis may benefit from further methodological advancements for more effective execution.

5. Conclusion

In this study, an evaluation of the FMS was undertaken with EDAS, and the objective weights were established through Shannon Entropy. With this method, alternatives for FMS could be systematically evaluated using data, making the decision-making process strong and dependable. A comparative review was made between the rankings and those produced by the COPRAS and SWEI methods to check their reliability. The analysis proved that both EDAS-Shannon Entropy, COPRAS, and SWEI rankings were consistent, which shows the methodology is solid and trustworthy. Moreover, a sensitivity analysis was done by adjusting the weights given to each criterion to see the ranking's consistency. The analysis showed that the rankings were mostly the same despite changes in the scenario, while only small shifts in some alternative positions were spotted. Such results demonstrate that the EDAS-Shannon Entropy method can provide reliable help for choosing FMS. The study offers helpful suggestions for people responsible for production and planning. The production systems are very important for producing the same types of products in the same quantity. Many manufacturing firms adopt flexible production systems to take advantage of the benefits provided by FMS. Many criteria are taken into account when selecting or evaluating any production system. Also, these criteria are sometimes confused with bias. Therefore, it is appropriate to use MCDM methods in FMS selection and evaluation. EDAS method ranks these systems according to the performance they show in the criteria of flexible production systems. The analysis of choosing FMS using the integrated EDAS and Entropy methods yields several key recommendations to improve decision-making and guarantee optimal results. The Entropy technique provides an impartial, data-driven approach to criterion weight assignment by examining the inherent variability (unpredictability) in the criteria. This ensures that more important factors receive proper attention, leading to more accurate and dependable results grounded in real data. Combining the Entropy method for objective weighting with the EDAS approach ensures a comprehensive and thorough evaluation of all FMS alternatives. This integrated method eliminates potential subjective biases and guarantees a well-rounded assessment of every option, thereby improving the selection process's accuracy. By following the above objective and comprehensive steps, organizations can make more informed, balanced decisions, ultimately leading to the selection of a more strategic and successful FMS. The integration of the Entropy and EDAS methods offers significant advantages for future research and applications across various fields, providing adaptable, versatile, and rigorous frameworks for addressing complex, multi-faceted problems. However, there are some limitations of the study. The suggested approach presupposes that the input data used is entire, consistent, and precise. Noisy and missing data could interfere with the integrity of the results in real-life industry settings. The application of Shannon Entropy will give only objective weights depending on data variability that might not fully represent the preferences of experts or priorities of managers in decisions in a practical situation. The methodology has been formulated in deterministic circumstances and makes no allowance for uncertainty or vagueness in the value of the criteria. This confines its use in the world where linguistic or imprecise data tends to prevail. Although the method is compared to COPRAS and SWEI to determine robustness, it would be worth greater validation on a wider scale by comparing the method against other hybrid or fuzzy MCDM approaches (e.g., TOPSIS (Technique for Order of Preference by Similarity to Ideal Solution), VIKOR, fuzzy EDAS). The framework takes a certain case of evaluation of FMS. Wider generalization might necessitate testing in several spheres or in an industrial environment of different complexity and set of requirements. Finally, Entropy weights are only based on the current data and thus fail to dynamically adjust to changing manufacturing situations or dynamic decision environments that could be factors in the flexible manufacturing context.

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Can student attitudes toward immigration be changed? Evidence from a survey experiment in Croatia

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Abstract. Extreme right-wing parties are increasing in polls around Europe, largely fueled by an anti-migrant rhetoric. Political economy literature points to, on average, net positive effects that migrants bring to the economy, but the balance on the political market is more worrisome. For a small open economy, overly dependent on tourism, whose population reduced by more than 1 million in the last 30 years, the question of successful integration of migrants represents a first order condition of public policy. Thus the research question set in this paper is how to change attitudes on immigration among students in the Croatian society. Our approach is based on an experiment within a survey and it is tested on a sample of 1,450 students from five university cities in Croatia (Osijek, Pula, Rijeka, Split and Zagreb). Results indicate that there is a sizeable and statistically significant effect for the treated groups vis-a-vis their attitudes on the effect that migrants have on the labour market, social security system, overall safety and the economic development of Croatia.

Keywords: behavioural political economy, experiment, immigration, students, survey analysis

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1. Introduction

"An idea is like a virus, resilient, highly contagious. The smallest seed of an idea can grow." (Inception)

The number of immigrants to Croatia is estimated to have quadrupled since 2018 according to numbers from the Croatian Pension Insurance Institute and the Ministry of Interior. According to estimates from the Ministry of the Interior [25], in 2024, there were close to 207 thousand working permits issued for citizens born outside of Croatia, up from 170 thousand issued in 2023, and up from only 70 thousand issued annually in earlier years. If these trends continue, it is evident that Croatia has entered a wave of immigration which could have overarching consequences on the Croatian society. Yet, little academic literature is available on the determinants of preferences on immigration and how these could be changed in Croatia. This article aims to fill this gap.

Existing international literature has extensively studied both the causes of preferences on immigration and the consequences of large immigration shocks on political preferences and economic outcomes of native populations. A large body of literature examines the effects of immigration on wages, unemployment rates and fiscal deficits in receiving countries. When studying the Miami boat lift that led to a sudden immigration wave that increased the workforce

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by 6%, Card [6] showed no discernible effects on wages nor employment rates of the native population. Overall, studies [21, 7] across a wide variety of contexts and countries, find very little or no impact of immigrant workers' arrival on natives wages. This impact is likelier to be null or positive if the skills of the immigrants are complementary rather than substitutory to the skills of natives. On the other hand, the strand of literature in political economy has found large effects on changes in political preferences and social cohesion in society. The political economy literature has demonstrated evidence that immigration waves have led to the popularisation of the extreme right parties across Europe that have exploited the salience of the topic to profile themselves as anti-immigrant [20, 22]. Beyond these changes in political preferences, immigration shocks, including the arrival of a large number of asylum seekers, have led to an increase in hate speech and hate crime [12, 28].

This research asks two questions. First, it descriptively asks which individuals are in favour or against immigration into Croatia among the student population. Within this research question, we specifically focus on understanding how individual characteristics such as their education, income level and age correlate to their views on the desirability of immigration to Croatia. Our second research question asks whether these negative attitudes on immigration can be changed through information nudging. Within this research question, we explore whether and which incentives (economic versus historical-compassion) affect respondents' attitudes. From the above research questions, we derived two hypotheses. H1: Information nudging based on scarcity in the labour market positively affects economics and business students' attitudes on immigration. H2: Information nudging based on historical experience and compassion positively affects economics and business students' attitudes on immigration.

For the purpose of answering these questions, we collected novel survey data on a sample of 1,450 business and economic students from five university cities in Croatia (Osijek, Pula, Rijeka, Split and Zagreb) in 2024. The survey consisted of thirteen questions across a series of socio-demographic and attitudinal questions. In the first part of our research, we rely on descriptive statistics and a linear regression estimated with ordinary least squares to understand covariates that predict immigration attitudes. Our second approach is based on a survey experiment. "A survey experiment is the deliberate manipulation of the form or placement of vignettes in a survey instrument, for purposes of inferring how public opinion changes" [15]. It implies random assignment of respondents to the control and treatment groups which allows the differences in preferences and behaviors of respondents between the two groups to be interpreted as causal. We construct two treatments - one information vignette describing the El Shatt refugee camp, and, another information vignette describing the lack of workforce in the Croatian labour market. While this methodology has been used in the research on immigration attitudes (for a review see [19]), our contribution lies in its, to the best of our knowledge, novel application in Southeast Europe. Moreover, this new national context, whose recent history is characterised by experiences of war-induced displacement, allows us to make a theoretical contribution by juxtaposing the effects of information targeting an economic justification versus a compassionate justification, building on the population's own collective memory of displacement and forced migration.

Descriptive results suggest that very few (one in ten or one in twenty) young individuals think that immigration to Croatia is a positive development. It seems this is not driven directly by economic concerns of immigrants taking their jobs or becoming a financial burden to society as many more individuals do not raise this as a concern. Examining correlates of immigration attitudes, we find that attitudinal variables predict attitudes on immigration well, while socio-demographic variables of individuals do not predict their attitude on immigration. Across almost all the outcomes, individuals who report that they believe all people are trustworthy are likelier to also have positive views on immigration. On the opposite side, individuals who report to be religious consistently are associated with a lower likelihood of accepting immigration. The findings from our survey experiment show the potential of information updating in changing views on immigration, if focused on a specific concern around immigration. Results indicate that there is a sizeable and statistically significant effect for the treated groups vis-a-vis their attitudes on the effect migrants have on the labour market and the social security system. This effect is stronger for the information vignette on workers' shortage. The information vignette discussing labour market shortages was more effective at changing attitudes on the economic effects of migration - namely reducing the number of individuals who believe immigrant workers will take natives' jobs or become a financial burden on the social security system.

The paper is organized as follows. Section 2 discusses the literature review and how this article contributes to it. Section 3 describes the data and research methodology while Section 4 discuss the results. Section 5 concludes this article.

2. Literature review

This research speaks to two strands of literature within political economy. First, it contributes to the literature on the political and economic causes and effects of immigration on society. A common starting point in the literature is that the domestic population misperceive both the size and the effects of immigration [1, 29]. Second, methodologically it contributes to the literature that uses survey experiments to advance our knowledge of the determinants of preferences on immigration.

An extensive literature has discussed the channels through which individuals form opinions on immigration, which can be summarised across two juxtaposing channels - one formed around political and economic concerns, and the other formed around social and cultural concerns. If political and economic concerns are important for our view on immigration, then citizens form attitudes about immigration based on its effects on their personal economic situation. For example, if citizens believe immigrants would depress their wage or 'take' their job, they are likelier to be opponents of immigration. However, empirical literature has not found support for such assumptions. For example, Hainmueller and Hiscox [18] discuss how immigration that changes individual's competitiveness on the labour market does not predict immigration attitudes. Empirically, this is supported by the lack of correlation between individuals' own education levels (whether less or highly educated) and their attitudes toward immigrants of differing education levels. Crepaz and Damron [10] also show that comprehensive welfare states are associated with lower nativist sentiment across the EU, where plausibly, if the hypothesis that concerns of immigrants becoming a burden to the social security system were true, we would expect the opposite direction of the association. Yet, the literature on the social and psychological reasons behind the formation of immigration attitudes has found a series of determinants. Overall it seems immigration-related attitudes are driven by concerns about "national identity": specifically either the (lack of future) cultural assimilation [33, 17] or concerns driven by ethnocentrism and stereotyping [23].

A special and interesting research avenue within political economy of migration are electoral effects of immigration. Immigration fuels support for populist radical right parties regardless of the absence of its negative impact on employment or political economy outcomes [2]. On a political market immigrants matter less for what they are and more for what they mean: cultural concerns and anxieties about identity and the "loss of control over our way of living". Increased news coverage (salience) rather than quality substance often drives electoral outcomes towards radical political options and towards polarization of the electorate as reported in France [31]. Informational distortion, carried out by social media and news, regarding the true number of immigrants in Italy [3] or "density stress" rhetoric – a narrative emphasizing overcrowding and social saturation contrary to the reality – in Switzerland [2] represent examples of "successful" strategies that demonise immigrants in the eyes of local and national electorates. Barone et al. [4] report several mechanisms that channel natives fear (labour market, public services and cultural diversity) into support for centre-right parties, particularly those with anti-immigration stances in Italy. Adding a novel twist to the demographic-political nexus, Dancygier et al. [11] argue that both trends, immigration of foreign workers and emigration of natives, change the composition of electorates as well as the political preferences of natives due to emotional grievances about being left behind. Together, they add to right-wing populism's success at the subnational level across Europe. With respect to Croatia and its electoral cycles the topic of immigrants has still not surfaced while the interest of the scientific community for overall migration topics is present and correlated, albeit with a time lag, with immigration trends. Namely, during the "migration crisis" of 2015-2016, when more than 650,000 migrants passed through the Croatian part of the Balkan route, respondents actively demonstrated a high level of solidarity with migrants who basically just passed through Croatia without an intent to stay and settle here [8]. Several years later the situation has evolved as it was documented on all levels within the society. Research that captured high school students' attitudes toward immigrants, based on Croatian subsample of students and their parents from the 2018 PISA survey, reports that attitudes towards immigrants are transmitted primarily through family socialization and online sources [30]. In a similar study to ours with respect to the sample, Mrakovčić et al. [27] report results of two rounds of surveys (2019 and 2024) on attitudes of students at four law faculties in Croatia towards migrants, foreign workers and asylum seekers. Social distance increased towards all groups of potential immigrants, except towards immigrants from the countries of the former Yugoslavia. In both rounds, value orientations were the strongest predictor of expressed social distance towards migrants and in the 2024 survey, as opposed to 2019 survey results, respondents perceived foreign workers as "undesirable". Focusing on the whole society and using data from the fourth, fifth, ninth and tenth rounds of the European Social Survey, in which Croatia participated, Medimurec et al. [26] identify four societal subgroups with respect to immigration attitudes. The study reports that two groups (pro- and moderate pro-immigration) have a majority within the society and that the share of the population with the most negative attitudes on immigration declines slightly over time. Using the same ESS dataset Čaćić-Kumpes et al. [8] report right-wing political orientation, together with low levels of education and religion as the main causes of respondents' negative stance toward migrants whose number significantly increased between 10th (2020-2021) and 11th (2023-2024) wave of ESS surveys in Croatia. The respondents emphasized cultural and not economic threats when asked what is their main concern regarding immigrants. Focusing on economic aspect of immigrants' life in Croatia, Butković [5] concludes that the success of their integration will depend on the type of services that they provide (intellectual vs. physical labour) and the role of digital platforms in their careers. Emphasizing that managing migration is a complex task Gregurović et al. [16] drew attention to the possible implications and challenges of the long-term settlement of foreign workers in Croatia. Since the goal of their paper is to provide an overview of the immigration of foreign workers to Croatia in the last two decades, our goal is to empirically test whether we could address economic concerns that should be of an interest for students of economics and business in the five university cities in Croatia. Moreover, the aim of this paper is to test whether attitudes can be changed and whether information targeting an economic justification has differential effects to information targeting a compassion justification. In the current climate, these questions are crucial political economic issues that deal with social spending, optimal taxation and determine the boundary between the activities of the state, companies and migrants in the integration process.

3. Data and research design

3.1. Data

This research is based on survey data collected among 1,450 business and economic students from five university cities in Croatia (Osijek, Pula, Rijeka, Split and Zagreb) in 2024. Through surveying random groups (grouped by the alphabetical order of their surnames) from different years and subjects, the research includes individuals with different socio-economic backgrounds and educational interests across all geographic regions of Croatia. The survey was short and only required 5-10 minutes to complete, which led to a negligible attrition rate of respondents that opened the survey in the first place. There were in total 1,450 respondents which determines our sample size. The survey experiment was conducted within the Qualtrics program. Students were informed that participation in the survey was voluntary and presented with an informed consent description before taking the survey. It described the purpose and the topics of the questionnaire, anonymity and data collection procedure, and the length of the survey to the participants who could voluntarily take part in it. Our research sample is broadly representative of the population of undergraduate students in Croatia, as the distribution of key characteristics closely matches population patterns. The gender profile (65% in the sample vs. roughly 60% in the population), the age profile (mostly 21–23) and regional composition (predominantly Central and Northern Croatia) align well with national data. Parental education is also similar: in both the sample and the population, parents most commonly have a high-school degree. It is also important to note that in the field of behavioural economics students are considered a usual subject pool as research is often conducted at universities and students are easy to recruit and motivate, they are computer literate and willing to take part in the research for non-monetary reasons [13, 14, 24].

There are five key questions in our survey that measure attitudes on immigration. They are phrased in the following way:

1. Immigrants are taking jobs from Croatians. "Please indicate your level of agreement with the statements below from zero to five (highest)"

2. Immigrants make the fight against crime worse. "Please indicate your level of agreement with the statements below from zero to five (highest)"

3. Immigrants represent an additional cost to the social security system. "Please indicate your level of agreement with the statements below from zero to five (highest)"

4. What would you say is the influence of immigrants on the development of Croatia? Responses are: 1 "strongly negative" 2 "negative" 3 "neither positive nor negative" 4 "positive" 5 "strongly positive"

5. The arrival of a larger number of immigrants to Croatia should be made possible. "Please indicate your level of agreement with the statements below from zero to five (highest)"

We additionally asked a series of socio-demographic variables. These included the age, date of birth, sex, whether the individual is working (categorical variable that included answers of working part time, and/or working while studying), their father's and mother's education levels, whether they believe their family was better/worse/equal off financially as compared to other families, and the region of their origin. We asked three attitudinal variables: i) whether the individual believes people are trustworthy, ii) whether the individual is religious, and iii) how the individual situates themselves on values around abortion, divorce, death penalty and other norms.

3.2. Survey experiment

Within the survey, we ran a survey experiment with two treatments and a control group. Individuals were randomly assigned to either the control group or the treatment group on the basis of the month they were born in. Once randomly assigned through the survey, each group saw a different information vignette before proceeding to answer five questions on immigration attitudes. The control group consists of 474 individuals, the first treatment group (El Shatt) of 493 individuals and finally the second treatment group (labour shortage) of 483 individuals. The balance in observables across the control and two treatment groups is shown in Table 1, where we show that there are no observable differences between these groups across variables such as gender, age, regional background, religiosity, and their father's education.

	Control: Mean (SD)	El Shatt: Mean (SD)	Labour: Mean (SD)
Female	0.65 (0.48)	0.64 (0.48)	0.66 (0.47)
Age under 23	0.94 (0.24)	0.91 (0.28)	0.93 (0.26)
North and Central	0.60 (0.49)	0.63 (0.48)	0.59 (0.49)
Religious	0.73 (0.45)	0.75 (0.43)	0.74 (0.44)
Father has a High School Degree	0.51 (0.50)	0.53 (0.50)	0.52 (0.50)

Table 1: *Balance table across treatment and experiment groups.*

The information vignettes are the following:

1. **El Shatt**: 250 words that describe the El Shatt refugee camp. El Shatt was a refugee camp in Egypt that housed 40,000 civilians mainly from Dalmatia (Croatia) that were fleeing the German offensive in the fall and winter of 1943–1944. The civilians and many children lived there for two years until the end of the war and their repatriation in 1946. The topic of this information vignette was chosen to incite empathy and compassion with immigrants as this episode in history represents an occasion when Croatians were on the receiving end of another country's hospitality and is often met with fond memories from Croatians.

2. **Labour market shortage**: 250 words that describe the current shortage of labour and workers across Croatia and the negative economic consequences this lack of workers might have on the wider economy if it were to persist. The topic of this information vignette was chosen to encourage the realisation that immigrants are very needed in certain aspects of the Croatian economy.

3. **Control: History of university**: 250 words that describe how the Faculty of Economics and Business was founded at the University of Zagreb.

As our study was a survey experiment, we pre-registered the study on the Open Science Framework OSF Preregistration prior to conducting the analysis, which can be accessed on the following OSF link or under the following DOI.

Our estimation specification is the following which we estimate using Ordinary least squares (OLS):

$$Y_i = \alpha + \beta_1 \text{El Shatt}_i + \beta_2 \text{Labour Shortage}_i + \epsilon_i \tag{1}$$

where Y_i are the five outcome variables that have been dichotomised and measure immigration attitudes of individual i, El Shatt estimates the effect of being in the treatment group receiving that information vignette, and Labour Shortage estimates the effect of being in the treatment group receiving that information vignette.

4. Results

This section describes the results in two parts, first describing the correlates that explain positive views on immigration within the student population, and second, by discussing the results from the survey experiment.

4.1. What correlates to having positive views on immigration?

As a starting point, the aim was to give an overview of attitudes and preferences about immigration among the student population in Croatia. We analyzed the five questions on immigration attitudes described in detail in the data section dichotomising them to capture positive attitudes towards immigration (the equivalent of 'positive' and 'extremely positive' or values of 4 and 5 for the variables whose answers were on a 1-5 scale). The summary statistics of these binary variables are shown in Table 2. When we look at the means of having a positive attitude across different consequences of immigration, we observe very different answers across these aspects. While for example 43.5% of respondents think that immigrants will not take their job, only 5.5% believe more immigration to Croatia is desirable and 11% of respondents believe immigration has overall positive effects on Croatian society. As many as 34% think that immigrants do not cause an increase in crime and 23.5% think that they there are not a social and financial burden to society. These results suggest that very few (one in ten or one in twenty) young individuals think that immigration to Croatia is a positive development, yet this is not driven directly by economic concerns of those immigrants taking their jobs or becoming a financial burden to society as many more individuals do not raise such concerns. This points to the fact that at least for a significant share of citizens, cultural reasons and fear of a lack of assimilation might be hindering their approval of immigration to Croatia.

	Mean	SD	Min	Max
Immigrants do not take jobs	0.4351724	0.4959506	0	1
Immigrants do not cause crime	0.3413793	0.474336	0	1
Immigrants are not a burden	0.2331034	0.4229534	0	1
Immigration has positive effects	0.1124138	0.3159838	0	1
More immigration is desireable	0.0565517	0.2310637	0	1
Observations	1450			

Table 2: *Summary statistics of main variables describing immigration attitudes.*

Next, we examine how each of these measures of preferences on immigration is correlated to each other in Table 3. Perhaps expectedly, we find they are all positively correlated to each other, so that, for example individuals who think that immigrants do not take jobs or cause crime are likelier to think that immigration has positive effects on society and is overall desirable. Within these positive correlations, we do find higher within correlation coefficients between the two overall measures of attitudes (such that individuals who think immigration has positive effects on society are also much likelier to think immigration is desirable) and lower correlation coefficients among measures of overall effects of immigration and those zooming into an aspect of immigration consequences.

Finally, to understand immigration preferences we use the five attitudes on immigration as outcome variables and estimate a simple linear model using ordinary least squares with the following socio-demographic independent variables describing the respondent: sex, age, whether they are working, whether their parents were poorer than other families, whether their

	Immigration has positive effects	Immigrants do not take jobs	Immigrants do not cause crime	Immigrants are not a burden	More immigration is desireable
Immigration has positive effects	1				
Immigrants do not take jobs	0.216^{***}	1			
Immigrants do not cause crime	0.227^{***}	0.345^{***}	1		
Immigrants are not a burden	0.196^{***}	0.302^{***}	0.401^{***}	1	
More immigration is desireable	0.357^{***}	0.116^{***}	0.151^{***}	0.147^{***}	1

* $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Table 3: *Correlation matrix of migration preferences.*

father and mother have a college or vocational degree and higher, and which region they are from. We also add two attitudinal variables: whether the respondent believes people are in general trustworthy and whether the respondent is religious (regardless of the type of religion). The results are reported in Table 4. Overall, across the five attitudes on immigration, we don't find that the socio-demographic characteristics are good predictors of whether or not the individual has more positive views on immigration. While occasionally some coefficients are statistically significant, none are significant across the majority of outcomes. However, we find that attitudinal variables, conditional on the socio-demographic characteristics, correlate to attitudes on immigration. Across almost all the outcomes, individuals who report that they believe all people are trustworthy are 9 to 18 percentage points likelier to also have positive views on immigration. On the opposite side, individuals who report to be religious are consistently associated with a lower likelihood of accepting immigration, from around 5 to 12 percentage points across the five measures.

4.2. Survey experiment results

The aim of this research was to also test whether and how attitudes on immigration could be altered. We randomised the treatment, consisting of participants either being shown the vignette describing the El Shatt refugee camp or a vignette describing the shortage in the workforce in Croatia. As the treatment is randomly assigned, we can interpret the estimates from Equation 1 as causal. The outcomes shown below are measured after seeing the vignettes. The full results are reported in Table 5.

We do not find any significant effects of the vignettes on the attitudes that immigration has overall positive effects nor that more immigration is desirable, as shown in Table 5. This could be explained by ex-ante overall low support for immigration across these attitudes or a treatment whose salience is not strong enough to change overall attitudes.

We do find significant and positive effects of the labour shortage treatment on the perception whether immigrants take jobs or become a social and financial burden to the receiving country. We do not find that this vignette has an effect on the perception whether immigrants cause crime. These findings are suggestive that a treatment targeting concerns around the economic impact of immigrants can change the perception of immigration as an economic threat, namely whether the immigrant workers will take natives' jobs or become a financial burden on the social security system. Moreover, the treatment effects have a large magnitude - reading the information on the lack of workers in the labour market in Croatia increases the likelihood of believing immigrants do not take jobs by 5.7 percentage points and increases the likelihood of perceiving immigrants as not being a burden on the social security system by 9.6 percentage

VARIABLES	(1) Immigrants do not take jobs	(2) Immigrants do not cause crime	(3) Immigrants are not a burden	(4) Immigration has positive effects	(5) More immigration is desirable
Men	-0.016	-0.058**	-0.002	0.015	0.028**
	(0.028)	(0.027)	(0.024)	(0.018)	(0.013)
Sex(NA)	-0.052	-0.043	0.038	-0.026	0.094*
	(0.114)	(0.110)	(0.098)	(0.073)	(0.054)
Older than 20	0.006	0.015	-0.059***	0.001	-0.020
	(0.026)	(0.026)	(0.023)	(0.017)	(0.012)
Working	0.037	0.000	0.025	0.011	0.006
	(0.027)	(0.026)	(0.023)	(0.017)	(0.013)
Poorer parents	0.043	0.063	0.022	0.009	0.015
	(0.048)	(0.047)	(0.042)	(0.031)	(0.023)
Educated father	0.059**	0.002	0.025	0.027	0.021
	(0.029)	(0.028)	(0.025)	(0.019)	(0.014)
Educated mother	-0.009	0.002	-0.010	0.012	0.009
	(0.029)	(0.028)	(0.025)	(0.018)	(0.014)
Dalmacija	-0.001	0.014	0.043	-0.031	0.017
	(0.039)	(0.037)	(0.033)	(0.025)	(0.018)
Slavonija and Baranja	-0.011	0.027	0.083***	0.010	0.037**
	(0.037)	(0.036)	(0.032)	(0.024)	(0.017)
Istra and Primorje	0.184**	0.096	0.121	-0.034	-0.026
	(0.086)	(0.083)	(0.074)	(0.055)	(0.040)
Bosna i Hercegovina	0.055	0.016	0.059	0.008	0.019
	(0.055)	(0.053)	(0.047)	(0.035)	(0.026)
Other region	-0.174*	-0.129	-0.111	-0.046	0.015
	(0.102)	(0.098)	(0.088)	(0.065)	(0.048)
People are trustworthy	0.178***	0.089**	0.101***	0.113***	0.012
	(0.038)	(0.036)	(0.032)	(0.024)	(0.018)
Religious	-0.118***	-0.111***	-0.053**	-0.076***	-0.023
	(0.030)	(0.029)	(0.026)	(0.019)	(0.014)
Observations	1,450	1,450	1,450	1,450	1,450
R-squared	0.041	0.021	0.023	0.035	0.017

Standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Table 4: *Immigration preferences: socio demographic characteristics and attitudinal variables.*

points. In relative terms, this is a 13% increase for the former outcome and a 40% increase for the latter outcome which is an economically sizeable effect.

The El Shatt vignette overall did not change attitudes on immigration. The only exception is that the vignete positively affected the likelihood of perceiving that immigrants are not a social and financial burden. Unlike Čapo [9] we do not report the awakened solidarity with migrants due to, a rather distant, refugee past that some Croatian citizens experienced during WW2. There are several possible reasons for the lack of solidarity. First, we opted for a refugee experience during WW2 instead of a more recent refugee experience during the Homeland war

in 1990s. Second, the El Shatt vignette was rather a regional experience that affected people from Dalmatia. Third, since the 1990s and its independence, nationalist ideology has dominated Croatia [32] and the El Shatt exodus experience does not fit well within that framework and it is also questionable how familiar current students are with the story. A recent paper by researchers from the Faculty of Political Sciences shows that during the last 30 years, the positive image of Tito and socialist movement has actually decreased over time especially among the young, the so called democratic generation born after 1982 [34].
In sum, our research shows that attitudes are changeable, at least in the short run. Our findings point to the fact that effects of potential information campaigns should be directed at one specific concern against immigration where they might have stronger effects. Specifically, in the two examples of the information vignettes, the one discussing labour market shortages was more effective at changing attitudes on the economic effects of migration - whether the immigrant workers will take natives' jobs or become a financial burden on the social security system.

VARIABLES	(1) Immigrants do not take jobs	(2) Immigrants do not cause crime	(3) Immigrants are not a burden	(4) Immigration has positive effects	(5) More immigration is desirable
El Shatt Treatment	0.052	0.026	0.052*	0.006	-0.012
	(0.032)	(0.031)	(0.027)	(0.020)	(0.015)
Labour Shortage Treatment	0.057*	-0.002	0.096***	0.021	-0.001
	(0.032)	(0.031)	(0.027)	(0.020)	(0.015)
Observations	1,450	1,450	1,450	1,450	1,450
R-squared	0.003	0.001	0.009	0.001	0.001

Standard errors in parentheses
*** $p<0.01$, ** $p<0.05$, * $p<0.1$

Table 5: *Survey experiment results.*

5. Conclusion

This paper investigates how individual characteristics correlate to students' attitudes on immigration and, more importantly, whether these attitudes can be changed through information nudges. A survey analysis was conducted among 1,450 economics and business administration students of various majors and subjects from five university cities in Croatia. The regression results show that socio-demographic characteristics are not good predictors of immigration preferences, but attitudinal variables, conditional on the socio-demographic characteristics, correlate to attitudes on immigration. The empirical strategy, for the second research question, relied on a randomly assigned survey experiment with two treatments and one control group. Interestingly, the main result is that the labour shortage information treatment causes a significant change in attitudes about immigrants in the labour market and the social security system. We find no evidence that the El Shatt treatment – when Croats were refugees – had a causal influence on any of the five questions that measure attitudes on immigration. In a nutshell, economics and business students positively reacted to an economic incentive (scarcity in the labour market) while compassion and the exodus experience that their fellow citizens experienced had no statistical effect on their responses.

The main limitation of our study is already recognised within the field of behavioural economics: uncertain long term effects. Information nudges, even if detected in the short run, might not persist long enough to have an effect on long term attitudes that influence behaviour. Moreover, while the responses were anonymous, they could still display social desirability effects at the baseline.

Overall, this research points to the positive effects information nudges can have on changing attitudes towards migrants. If policymakers are interested in promoting a more open attitude toward migrants, an effective strategy is to identify their main fears related to migrants through surveys, and then with various behavioral interventions, including short-term nudges and long-term educational policies, use evidence-based empirical analysis to dismantle myths about the impact of immigration on various economic, social, and political outcomes in society.

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Doubly effective one round election model: Improving efficiency and fiscal savings

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Abstract. Electoral processes are usual in most countries around the world. The most dominant electoral system is the standard Two-Round System (TRS). Despite the variety of models for electing public representatives, all result in public expenditure. In democratic societies, we cannot avoid such an expense, but we can minimize it. Minimizing expenses in line with simplicity and justice of the electoral processes is the primary goal of this paper. We propose an entirely new one-round election model that can replace and successfully simulate the existing two-round model. Our proposed solution is unique, and we have named it the Doubly Effective One Round election model (DEOR election model). The literature contains many similar approaches, such as Ranked Choice Voting (RCV)/Instant-Runoff Voting (IRV), Single Transferable Vote (STV), Approval Voting, or Majority Judgment, but this proposed model is entirely new. As the new model is based on a ranking method, it offers additional advantages and possibilities, which are explained and illustrated through several examples in the paper. In addition to its efficiency, the application of this model can bring significant fiscal savings.

Keywords: ballot, election round, fiscal savings, public good

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1. Introduction

The motivation for this paper is to improve the efficiency of the election process and reduce public expenses, thereby creating fiscal space for other public investments and enhancing the availability of public goods. Because most election processes consist of two rounds, the primary goal is to have only one round. There are various options for single-round election models, but this solution is unique. Therefore, the main objective of the paper is to develop an electoral model that successfully simulates the classic two-round system in only one round. In this way, the model will theoretically address a gap in the existing literature and practice. Its implementation will increase efficiency and yield significant financial savings.

Election models or electoral systems are the foundation of parliamentary democracy. Douglas Rae [26], in his 1967 work "The Political Consequences of Electoral Laws" underlines this point and it has become one of the most renowned papers in the field of election models. A comprehensive overview of election systems is also given in Arend Lijphart [23] significant paper from 1994 in which the author offers a detailed analysis of 70 election systems from 27 states during the period from 1945 to 1990. Such a large sample allowed for more precise conclusions. One of the most significant findings is that election models in general are not as different from

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each other, nor as complex from the voters' perspective, as they may appear initially. The success of specific election models lies in the design of a combination of characteristics that provide a fair and justified compromise for all stakeholders [4].

Generally, there are many dimensions of election models that are the basis for differentiation between them. One of the simplest is the way voters' preferences are expressed. According to this, we may select a single candidate, use ranked choice, score voting/range voting, or approval voting; very rarely, there is negative voting or the closure of candidates. The Doubly Effective One Round (DEOR) model uses augmented rankings with multiple choice.

Election models can be distinguished by the number of election rounds. There is a one-round election model (as in our DEOR model) and a two-round election model. The third and fourth options involve iterative elections, which eliminate candidates in each round, and elections with multiple rounds. Election models can also be differentiated by the method of decision making after voting. We have a majoritarian system, requiring more than 50% of the votes; the second is pluralistic (first-past-the-post-elections); the third is proportional which allows votes to be distributed according to contribution (e.g. the d'Hondt method which is very well represented in Croatia); and the last is compromise decision-making. Our model combines a majoritarian model in the first round with compromise decision-making in a hypothetical second round. The type of election result is another criterion for differentiating election models. The goal may be to elect a single winner (as in presidential elections), multiple winners (as in parliamentary elections), or to produce a ranked list (as in internal party processes). Our suggestion is to elect a single winner or multiple winners. Election models can also be judged in terms of their resistance to manipulation, such as false rankings, changes in decision, or disloyalty.

Our DEOR model increases resistance with more possibilities of voter preferences. Electin models can also be distinguished by their transparency or the complexity of vote evaluation. They may be simple, complex or computer verifiable. The DEOR model uses simple mechanism that can be verified by both humans and computers. One of the criteria is the impression of fairness of election models. Our proposed model provides very precise voter positions by ranking politicians according to their preferences and simulates the second round of the two-round system (TRS) in the second evaluation step, without the need to re-express preferences, which enhances security.

This model does not concern institutional design or construction. This paper focuses on the most efficient solution for the materialization of voters' rights. The fundamental goal in the process of translating voters' preferences into election outcomes must be justice and fairness. Despite all varies and different variations between countries – such as political system, geographical dispersion, voter concentration, history, heritage, number of parties [24], level of democracy, election structure, rule of law, level of decentralisation, social structure, economic development, and other determinants – there are a significant number of solutions in election process policy. Literature from this perspective has become increasingly popular [12, 14]. According to the complexity of the problem with collective choice and social welfare, we can say that the main problem in a broad perspective with preferences and aspirations of voters is interdisciplinary [30].

After the introduction in the paper, we present existing similar election models. The third part elaborates an extended explanation of our proposal, a one-round model. The last part presents the results and discussion of the proposed model with concluding remarks.

2. Existing election models

In an average country in the world, elections are a democratic standard. The decisions voters make under political circumstances are preconditions for conducting election policies. While intentions may be positive, the outcome can be devastating. Therefore, it is important to be aware of both the positive and negative aspects, considering all their pros and cons, as well as the differences between the various models [28]. The most well-known model is Ranked Choice Voting (RCV), also known as Instant-Runoff Voting (IRV). The main presumption is that voters rank according to their preferences. If no candidate has a majority, the candidate with the fewest votes is eliminated, and their votes are redistributed to the remaining candidates until someone reaches a majority. Examples of such elections can be found in Australia, where they are used for elections of the House of Representatives, as well as in some cities in the USA, such as New York, San Francisco, and Minneapolis, and in Ireland for presidential elections. This election model increases voter participation and reduces the intensity of negative campaigns. Another important benefit is lower fiscal pressure on the budget [27].

The next model is the Single Transferable Vote (STV), which is a proportional system in which voters rank politicians. The elementary concept of a proportional system is that the distribution of politicians is the same as the distribution of views in the electorate. This model is predominantly used in countries with strong English influence [32]. The most well-known usage of STV is in Australia, Ireland, and Malta. This election model allows countries with large pluralities to offer the most appropriate candidates from a wide range of political options. Consequently, this differentiation of voters' preferences allows smaller political parties a more significant role in the political process and more chances for success [6]. Another consequence is lower wastage of votes, but on the other hand, it is harder to understand how it works in practice for an average voter. Minus is also a complicated system of vote counting. This model is extensively elaborated in Farell and McAllister [11] on Australia's example. Another explanation also gives the work of Farell and Katz from 2014 [10]. According to practice, STV can be manipulated despite the theoretical perspective, which says that STV is difficult to manipulate [35]. The most debatable fact is the complexity of the model, and it can be mistaken in vote counting, where analysis of mistakes has shown that those mistakes can have a significant influence on election results [5].

Approval voting is a simple model that allows voters to approve of politicians as many times as they want. Among all models, approval voting is the most genuine of all similar voting systems, and it is the only model that secures the politicians with a Condorcet majority if the preferences are dichotomous. The consequences could be strengthened for centrist politicians in the most dominant two-party system [7]. Voters can choose more candidates from a ballot where every chosen candidate gets one vote. There is no ranking. The candidate with the highest number of approvals is the winner. This model is appropriate for elections with more candidates. It eliminates vote splitting. An explanation of the model's mathematical perspective is given in the paper by Van der Straeten et al. [34]. Because there are significant differences between voters in honesty [36], some authors give a comparison of election results that use approval voting from the perspective of honest strategic voters. Approval voting advocates for more candidates with the same treatment. Results give a better voice to voters with lower costs at the end of the process [16]. The negative side of this model is the uncertainty of voters' candidates' selection because they are unsure whether to choose only one candidate or all candidates that are acceptable to them [33]. Analysis has shown that this model gives the best results in situations when there are not too many and not too few candidates who compete [25].

The crux of the Majority Judgment model is an entirely new theory of social choice where voters judge the politicians [2], and it avoids and differs from the well-known Condorcet and Arrow paradoxes [21]. Majority Judgment is an electoral reform that satisfies all the preconditions for a successful election model [21]. The Condorcet rule is based on repeated voting and elimination of one candidate in every round, which is the basis of consistency [1], but on the other hand it could be possible Condorcetov paradox which means that the winner candidate wins all other in the game of pairs based on majority rule [20]. The Majority Judgment model is more realistic and lifelike. Balinski and Laraki [2] believe that the question is not how to convert many individual rankings into one decision, but rather, after defining a common language of grades to measure merit, how to convert voters' wishes into one collective evaluation of all competitors. The basic presumption is to evaluate, not to compare [12]. Some disadvantages and weaknesses of this model are shown in [19]. In the literature, there are some recommendations to use multiwinner voting as the process in electing politicians [9], which can be a solution.

For future research, we explore some specifics for a similar RCV model. We chose RCV because it is the most popular among researchers, and has the most usable data, which could be a good path for the implementing our model in practice. Some research showed that Ranked Choice Voting (RCV)/Instant-Runoff Voting (IRV) could increase voters' participation in elections and voters' curiosity. In contrast, others stated in favour that just younger voters are attracted [8], but not the overall population [18]. The paper [30] shows that RCV increases the chances for women in the election process, especially minorities. [15] showed that RCV has a good impact on a better perception of voters on the legacy of elections and election justice in comparison with traditional election models. From the negative perspectives of RCV, [13] concluded the importance of ballot construction as one of the significant reasons for RCV imperfections. However, some researchers have found a solution to the challenge mentioned. The work of [31] investigates appropriate ways of ballot construction to avoid one of the main complaints about this method. They suggested the System Usability Scale (SUS). Such a solution could also be easily implemented in the DEOR method, and it is detailed explained in [22]. Current literature from this field and all comparable models, with their pros and cons, is a precondition for a practical approach to implementing our DEOR model based on normative and practical goals. The detailed explanation is given below.

3. Methodology

Considering elections where only one candidate from the election list needs to be selected, a very common option is a classical model with two election rounds. This is the case with all elections in the Republic of Croatia since its independence. Since the very first time, Croatia has held eight presidential elections, 13 parliamentary elections, and nine local elections. One of our main motivations is to reduce the fiscal costs associated with the election process, so we propose an entirely new election model here. That model includes just one election round and can effectively replace and simulate the existing two-round model, with additional advantages and possibilities. In elections where only one candidate from the list needs to be selected, we use a classical two-round model. This is the case with all elections in the Republic of Croatia since its independence. Since the very first time, Croatia has held eight presidential elections, 13 parliamentary elections, and nine local elections. One of our main motivations is to reduce the fiscal costs associated with the election process, so we propose an entirely new election model here. That model includes just one election round and can effectively replace and simulate the existing two-round model, with additional advantages and possibilities.

3.1. Classical two-round election model

This is a standard model for selecting a single candidate from the list, with the following rules. In the first round, each voter must pick only one candidate from the ballot. If he chooses more than one candidate or none, his ballot is invalid. If a particular candidate gets more than 50% of the total number of valid votes, the election process is completed. Otherwise, two candidates with the most votes enter the second round after a few days (or weeks), where the election process is repeated just for them.

Obviously, organizing and implementing this model requires significant financial resources. The considerable financial savings could be achieved with just one round instead of two. However, can the overall process efficiency be retained in that case? With appropriate modification of the voting method, the answer is positive.

3.2. New doubly effective one-round election model

The main idea behind the one-round election model is that voters rank the candidates offered rather than choosing one. This approach is quite natural and practical. Namely, choosing only one candidate means that all the others are discarded as worthless. But in reality, that is never the case; everyone has their pros and cons. Each voter, according to his own perception, creates his own mental ranking of the candidates. This model enables the voter to copy it to the ballot. In this way, the ballot represents his real perception of all the candidates and not only one of them.

Technically, the situation is straightforward. If we have n candidates, then each candidate on the ballot is associated with n numbers 1, 2, 3,$\ldots$, n that the voters can choose. For each candidate, the voter has three possible choices:

- to choose only one number (number 1 if this candidate is his first option, two if he is the second option, etc.),
- to choose two or more numbers if this candidate is one of his multiple options (e.g. 1,2 and 4 if he is the first, the second, and the fourth option),
- not to choose any number if this candidate is by no means his option.

We can see how these choices enable the voter to express their opinion of each candidate in relation to the others. In this way, the ballot represents his complete perception of the election list. Note that this way of filling out the ballot is very natural and familiar to the voter.

In this way, the election process is completed in one round. In order to simulate the usual two rounds, the ballots are evaluated in two rounds.

In the first round of evaluation, valid ballots are those in which only one 1 is selected. If more than one or no number 1 is selected, then such a ballot is invalid in this round. This fully corresponds to the case in the classical model when more than one or no candidate is picked out. Thus, number 1 selected for a particular candidate makes a valid vote for this candidate in the first round, if number 1 is not selected for any other candidate on this ballot. Similarly, as above, if someone receives more than 50% of the total valid votes, the election process is completed, and the second evaluation round is held for the two candidates with the most votes. Such validation makes the first round identical in both models.

In the second round of evaluation, the valid ballots from the first round that support the two candidates who advanced to the second round are transferred to the second round. The other ballots are evaluated so that only the selected numbers for these two candidates are considered. The vote goes to the candidate with a smaller selected number. Valid ballots in this round are those with different selected numbers for these two candidates, or with a selected number for only one of them. Invalid ballots are now those with exactly the selected numbers and no selected number for both candidates. Let us remember that a voter can select more than one number for the same candidate. In such a case, the smallest one is considered. Now, the votes for these two candidates, from valid ballots in the first and second rounds, are counted to determine the election winner.

4. Results and discussion

The method of two evaluation rounds in a one-round election model, which we have just explained, will be illustrated through several examples. Suppose we have six candidates, A, B, C, D, E, and F, in the election. The ballot can be structured as follows.

A	...	1	2	3	4	5	6
B	...	1	2	3	4	5	6
C	...	1	2	3	4	5	6
D	...	1	2	3	4	5	6
E	...	1	2	3	4	5	6
F	...	1	2	3	4	5	6

Figure 1: *Six candidates ballot.*

Let us suppose that some voters make the following selections.

A	...	1	2	3	4	5	6
B	...	①	2	3	4	5	6
C	...	①	2	3	4	5	6
D	...	①	2	3	4	5	6
E	...	1	2	3	4	5	6
F	...	1	2	3	4	5	6

(a) Ballot 2

A	...	1	②	3	4	5	6
B	...	1	2	③	4	5	6
C	...	1	2	③	4	5	6
D	...	1	2	③	4	5	6
E	...	1	2	3	④	5	6
F	...	1	2	3	4	5	6

(b) Ballot 3

Figure 2: *Ballots 2 and 3 selections.*

A	...	1	2	3	4	⑤	6
B	...	1	2	③	4	5	6
C	...	①	2	3	4	5	6
D	...	1	②	3	4	5	6
E	...	1	2	3	4	5	⑥
F	...	1	2	3	④	5	6

(a) Ballot 4

A	...	①	2	3	4	5	6
B	...	1	2	3	4	5	6
C	...	1	2	3	4	5	6
D	...	1	2	3	4	5	6
E	...	1	2	3	3	5	6
F	...	1	②	3	4	5	6

(b) Ballot 5

Figure 3: *Ballots 4 and 5 selections.*

A	...	①	2	3	4	5	6
B	...	①	2	3	4	5	6
C	...	1	②	3	4	5	6
D	...	1	2	③	4	5	6
E	...	1	②	3	4	5	6
F	...	1	2	③	4	5	6

(a) Ballot 6

A	...	1	2	3	4	5	⑥
B	...	1	②	3	4	5	6
C	...	1	2	3	4	5	6
D	...	①	2	3	4	5	6
E	...	1	2	3	4	⑤	6
F	...	1	2	3	④	5	6

(b) Ballot 7

Figure 4: *Ballots 6 and 7 selections.*

A	...	1	2	③	④	5	6
B	...	①	②	③	④	⑤	⑥
C	...	1	2	3	4	⑤	⑥
D	...	1	2	③	4	⑤	6
E	...	1	2	③	④	⑤	⑥
F	...	1	2	3	4	5	6

(a) Ballot 8

A	...	①	2	3	4	5	6
B	...	①	②	3	4	5	6
C	...	①	2	③	4	5	6
D	...	①	2	3	④	5	6
E	...	①	2	3	4	⑤	6
F	...	①	2	3	4	5	⑥

(b) Ballot 9

Figure 5: *Ballots 8 and 9 selections.*

Let us suppose that no candidate obtains more than 50% of the valid votes in the first evaluation round, and that candidates C and E receive the most, say 40% and 35% (note that all the ballots with exactly one selected make 100%). They bring their votes to the second evaluation round, where they get additional votes from the remaining ballots. C (E) gets an additional vote if its selected number is smaller than E (C) or with any selected number if no number for E (C) is selected. We will now evaluate the given ballots 1-9 and compare them with the corresponding cases in the classical election model.

Ballot 1 is invalid in both evaluation rounds because it is empty, and no number is selected. This is the same as in the classical model with an empty ballot.

Ballot 2 is invalid in the first round because multiple number 1 is selected. It is valid in the second round where C and E are considered, and it provides the vote for C because number 1 is selected for C and none for E. Equivalent situation in the classical model is that the voter picks out three candidates in the first round and thus makes his ballot invalid, while in the second round he votes for C.

Ballot 3 is invalid in the first round because no number 1 is selected, but it is valid in the second round, where it provides the vote for C (the smaller selected number between C and E). An equivalent situation in the classical model is that the voter does not vote in the first round, or he leaves an empty ballot, and in the second round he votes for C.

Ballot 4 is valid in the first round, where it provides the vote for C. Since C passes to the second round, this vote is just kept for C in the second round. Equivalently, in the classical model, the voter gives his vote to C in the first round and in the second round again.

Ballot 5 is valid in the first round, where it provides the vote for A. Since A does not pass to the second round, this ballot is invalid in the second round because no number for C or E is selected. Equivalently, in the classical model, the voter gives his vote to A in the first round and does not vote in the second one or leaves the ballot empty.

Ballot 6 is invalid in the first round because number 1 is selected twice. It is also invalid in second round where the same number 2 is selected for C and E. Equivalent situation in the classical model is that the voter makes multiple choice in the first and then in the second round again.

Ballot 7 is valid in the first round, where it provides the vote for D, but D does not pass to the second round. The ballot is also valid in the second round where it provides the vote for E because number 5 is selected for E and none for C. Equivalently in the classical model the voter gives his vote to D in the first round and to E in the second one.

Ballot 8 is valid in the first round, because of one selected number 1, where it provides the vote for B, but B does not pass to the second round. The ballot is also valid in the second round, where it provides the vote for E because the smallest selected number for C is 5, and for E is 3. Equivalently in the classical model the voter gives his vote to B in the first round and to E in the second one.

Ballot 9 is invalid in the first round because of multiple selections of number 1, and also in the second round because of the same smallest selected number (number 1) for C and E. Equivalent situation in the classical model is multiple choice in the first round, and then again in the second round.

The main idea was that DEOR simulates TRS as well as possible, and that is why the evaluation rules are like that. Such an approach also has certain disadvantages. For example, let us suppose that 90% of voters equally prefer candidates A, B, C, D with number 1 and they ignore candidates E, F. Then their ballots are not taken into account both in the first and in the second evaluation step and the remaining 10% of voters can choose winner E or F, although 90% of voters do not want that candidate E or F wins. An equivalent situation occurs in TRS when 90% of voters make multiple choices for A, B, C, and D. Of course, the probability of such a situation occurring in reality is practically zero.

Under different evaluation rules, e.g., a rule that in the first evaluation step each of candidates A, B, C, D gets $\frac{1}{4}$ vote from each ballot (in total each of them gets 22.5% of the votes), then neither E nor F can be a winner. Such changes can improve the model, but we lose equivalence with TRS. Modifying and optimizing the evaluation rules to improve the model can be an interesting topic for future research. We summarize here the properties and advantages of the proposed model in relation to the classical one.

As we have seen, the classical two-round election model could be replaced entirely with the proposed one-round model. It means that the same effects obtained from two rounds can be achieved just from one round. There are significant savings in time and money in the organization, administration, and conduct of the election due to a single round rather than two. There are also financial savings in the election campaign, as it is held only once. To improve efficiency in such circumstances, the campaign should be more serious and thorough, and its quality raised to a higher level. The election ranking method makes a ballot an image of the voter's detailed perception of all the candidates. It can be a background for valuable post-election analysis of public opinion. In the classical model, if a voter gives his vote to a candidate who advances to the second round, he can change his mind and cast his vote for another candidate in the second round. This is not possible in the one-round model. The candidates who pass to the second evaluation round bring their votes from the first round. This feature encourages voters to think carefully about their choice, and candidates to conduct their campaigns thoughtfully. If the voter chooses a particular candidate, then they should be consistent in their choice. However, the paper based on five presidential French elections shows that voters have different preferences between two rounds [3]. Finally, as we have seen in the presented examples, if a voter has any doubts and cannot make a firm decision (as the classical model forces him to do), he can express minor differences in his perception of the candidates through the ranking method in the proposed election model.

5. Conclusion

In the paper, we show that the classical two-round election model can be replaced by a one-round model. This can be achieved by introducing a ranking method rather than selecting a single candidate. Instead of conducting two rounds, the ballots are evaluated twice. The first evaluation simulates the first round and yields the winner or two candidates for the second evaluation. In this way, two physical rounds are replaced with two evaluation rounds. This results in significant savings in time and money with the same output. Besides, the Doubly Effective One Round election model (DEOR election model) enables the voters to express their perception of each candidate instead of choosing just one and discarding all the others. This can be an additional motivation to vote, as many voters miss the election due to doubts about the candidates offered.

In conclusion, the presented model is much simpler and more efficient than the existing one. It encourages candidates and voters to approach the election process more seriously and consistently. For subsequent research on this matter, there is room to explore the practical and experimental aspects of this new method, and we encourage this approach, as this paper is based solely on a theoretical perspective.

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Croatian Operational Research Review

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Operational Research is typically concerned with optimizing some objective function, or finding non-optimal but satisfactory solutions. It generally helps management to achieve its goals using scientific methods. Some of the primary tools used by operations researchers are optimization, statistics, probability theory, queuing theory, game theory, graph theory, decision analysis, and simulation.

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- Decision analytics,
- Econometrics,
- Linear and nonlinear programming,
- Inventory theory,
- Optimal maintenance,
- Scheduling,
- Stochastic process, and
- System analysis.

Operational Research has improved processes in business and all around us. It is used in everyday situations, such as design of waiting lines, supply chain optimization, scheduling of buses or airlines, telecommunications, or global resources planning decisions.

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